UCLA
Department of Mathematics

Spatial Random Processes and Statistical Mechanics
Oberwolfach Workshop ID 0636

Cardy's Formula for Certain Models of the Bond Triangular Type

Joint work with H.K. Lei

Talk Outline

I Background & Smirnov's Proof

II Triangular Bond Model

III Model Under Consideration

IV Path Designates

V Color Symmetry Without Conditioning

VI Color Symmetry Under Conditioning

VII Crossing Probabilities

VIII Summary of Technical Difficulties

IX Loop Erasure

Work of Smirnov takes place on the triangular site lattice, equivalently hexagon tiling of $\mathbb{C}$.

Hexagon yellow:  probability p.

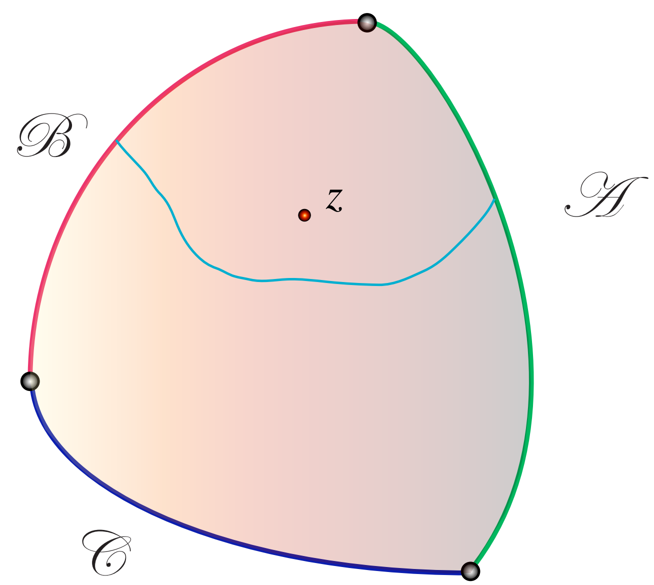
Hexagon blue:  probability $(1-p)$.

Critical point at $p_c = \frac{1}{2}$, i.e.

$p > p_c$ percolation of yellows

$p < p_c$ percolation of blues

Central Practical Goal

U(z): 

$$u(z) = \mathrm{P}\,(\mathbf{U}(z)),$$

then as lattice spacing tends to zero, $u(z)$ converges to an "appropriate" harmonic functions. Similarly define the functions v and w.

- With boundary and analyticity conditions (more later), u and related functions are uniquely specified (conformally invariant).
- The said harmonic functions are linear on the equilateral triangle and satisfy Cardy's Formula.

Key Ideas

I. Harmonic Triples (120 degree symmetries)

- $u + \frac{i}{\sqrt{3}}(v - w)$, etc. are analytic functions.

- 120 degree Cauchy-Riemann type equations like

$$D_{\hat{s}} u = D_{(\tau \hat{s})} v \quad ; \quad \tau = \exp(\tfrac{2\pi i}{3})$$

- Equilateral triangle: u, v and w are linear and do satisfy Cardy's formula.

Enough to show on an arbitrary domain u, v, w satisfy the same boundary/derivative conditions as on the equilateral triangle (solution to the same conformally invariant problem).

II. Lattice Functions

Boundary/derivative conditions seen from the lattice functions.

- Boundary conditions are easy and lattice independent.

- Main difficulty: Cauchy-Riemann Equations.

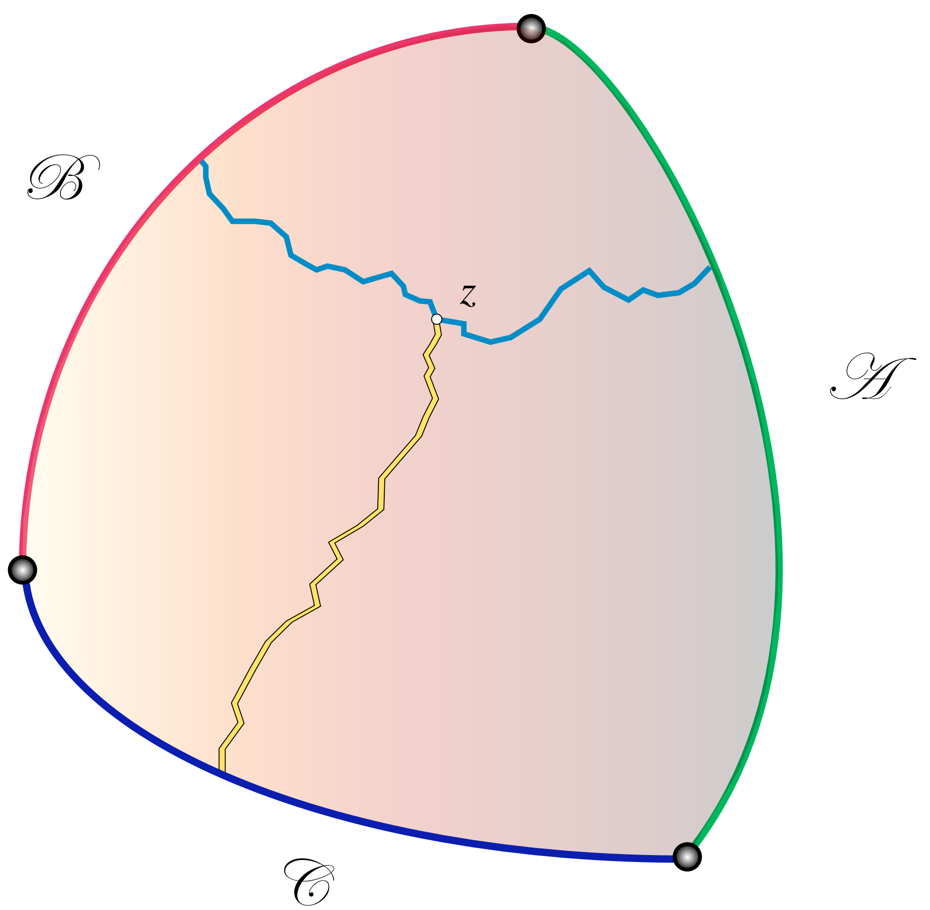
Discrete Derivatives and Color Switching

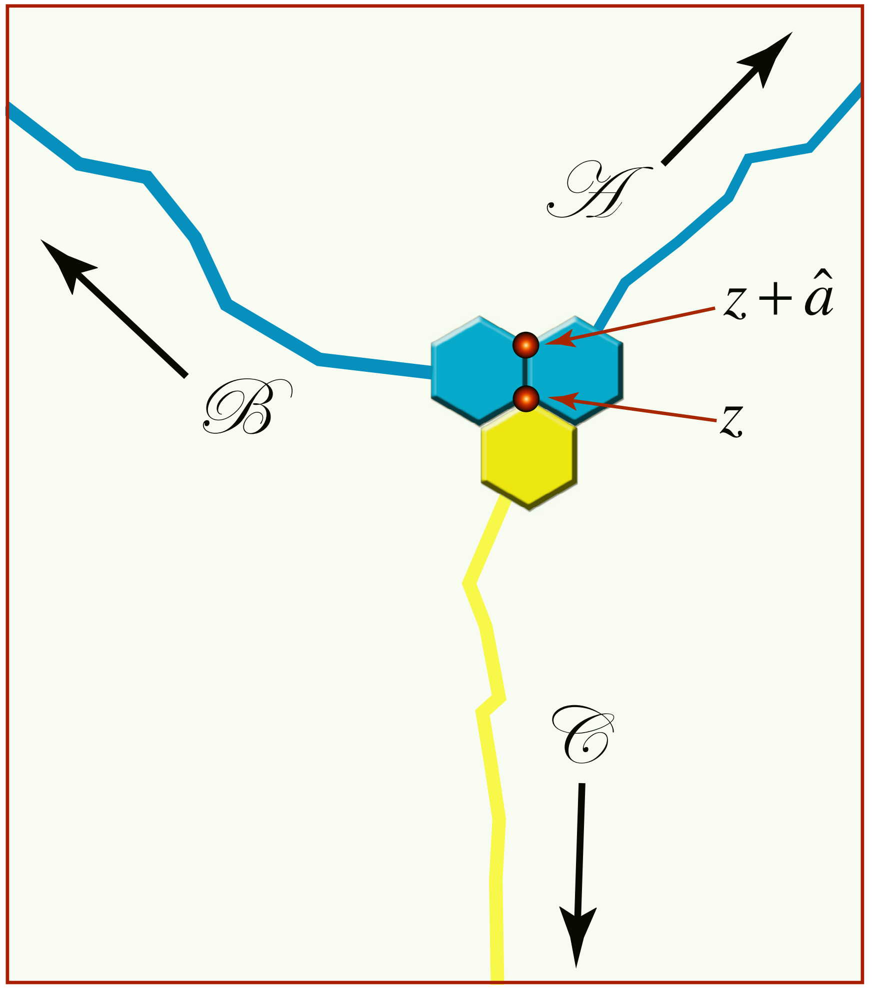
The discrete derivative is given by

$$u(z+\hat{a})-u(z)$$

which is seen to equal

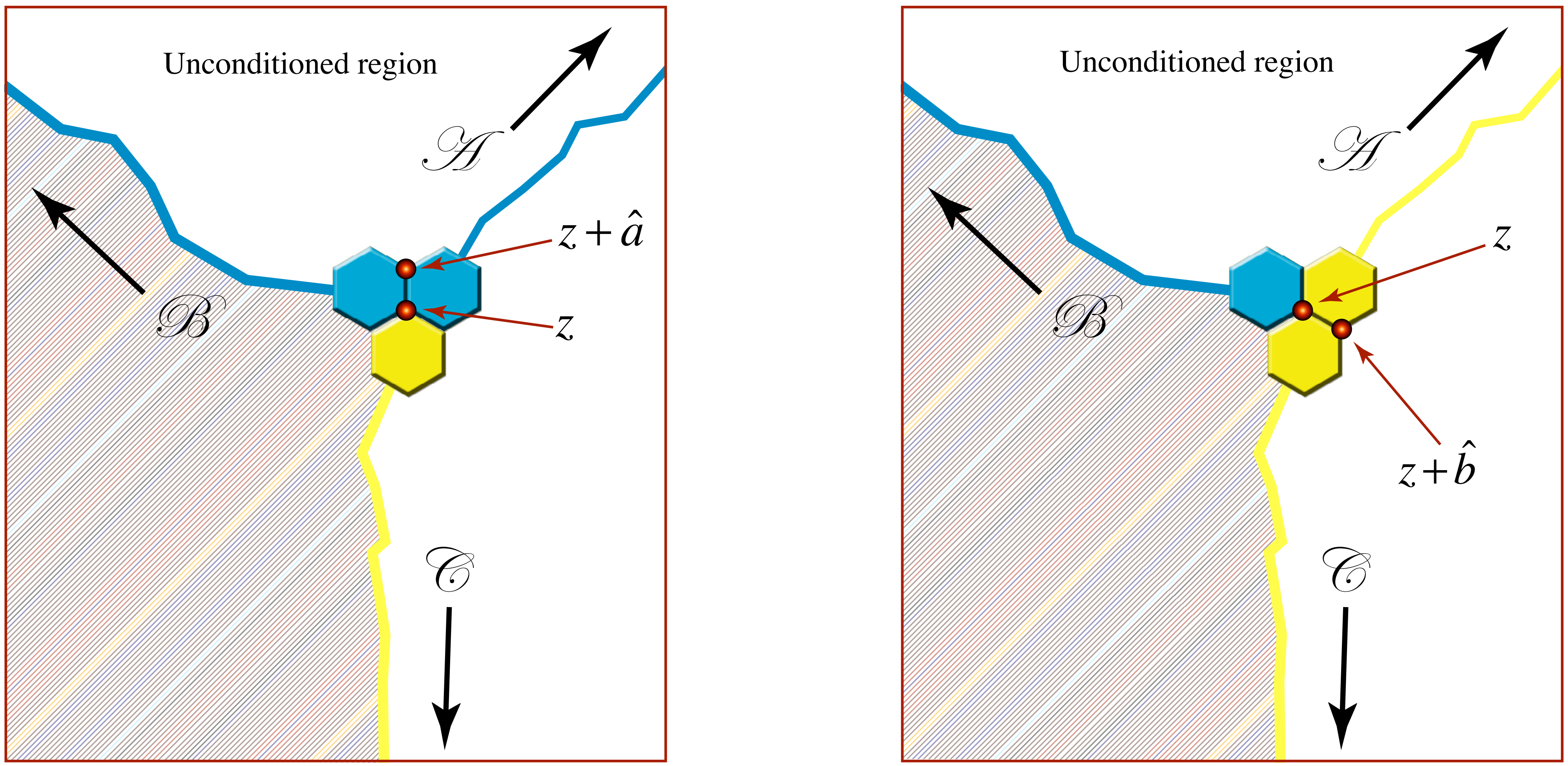
$$\mathrm{P}\,[\mathbf{U}(z+\hat{a}) \setminus \mathbf{U}(z)] - \mathrm{P}\,[\mathbf{U}(z) \setminus \mathbf{U}(z+\hat{a})] = \mathbf{U}_a^{+}(\mathrm{z}) - \mathbf{U}_a^{-}(\mathrm{z})$$





The CR relations: let $\hat{a}$ & $\hat{b}$ be two lattice vectors as shown, then

$$\mathbf{U}_a^+ = W_b^+$$

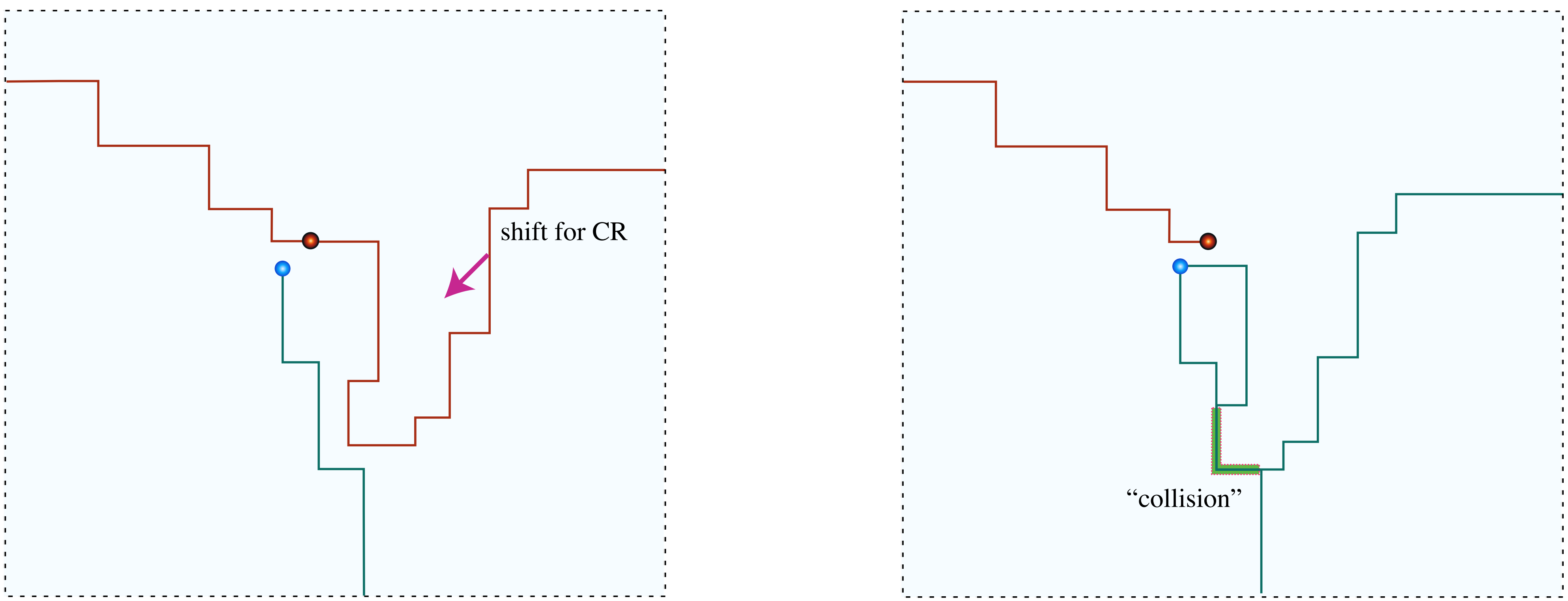


i.e. the probabilities of the two "CR–pieces" are the same.

This together with some analysis is enough to push through a proof.

Difficulties With Other Lattices

It is a miracle of the triangular site lattice that these innocuous looking CR relations hold without apology. E.g., on the square lattice:

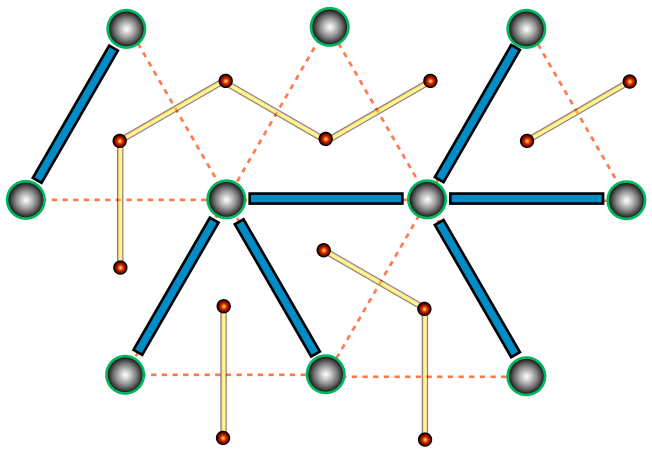


Here “collision” problems occur when trying to switch colors.

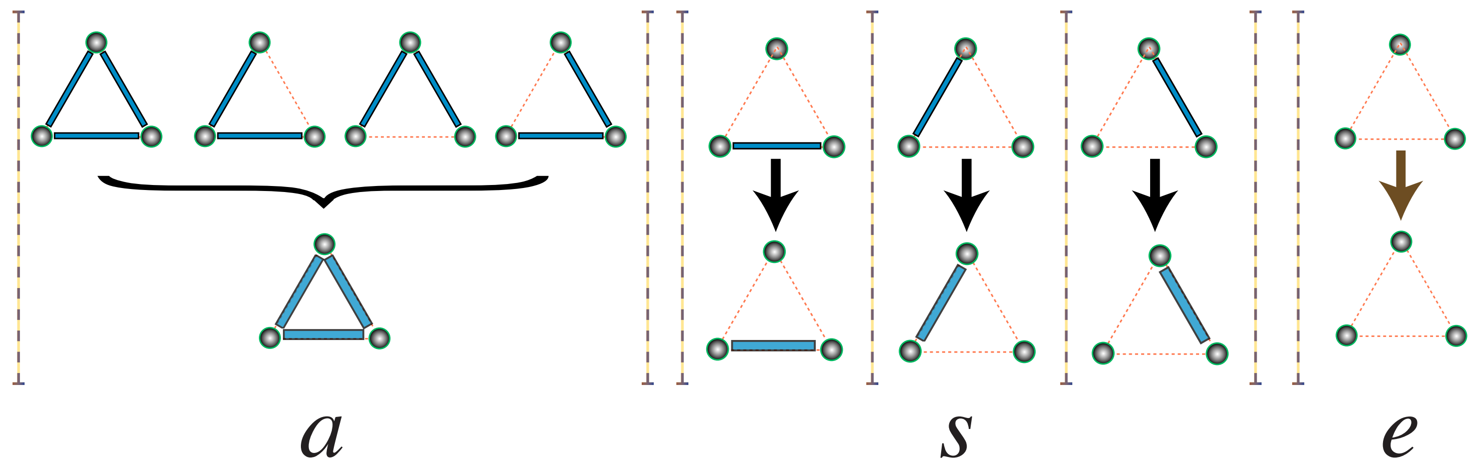
Model under consideration: Based on triangular lattice *bond* percolation problem.

L. Chayes and H.K. Lei, Random Cluster Models on the Triangular Lattice, J. Statist. Phys. **122** no. 4, 647–670 (2006).

(1) Bonds independently *blue*: p / *not–blue*: $(1-p)$.

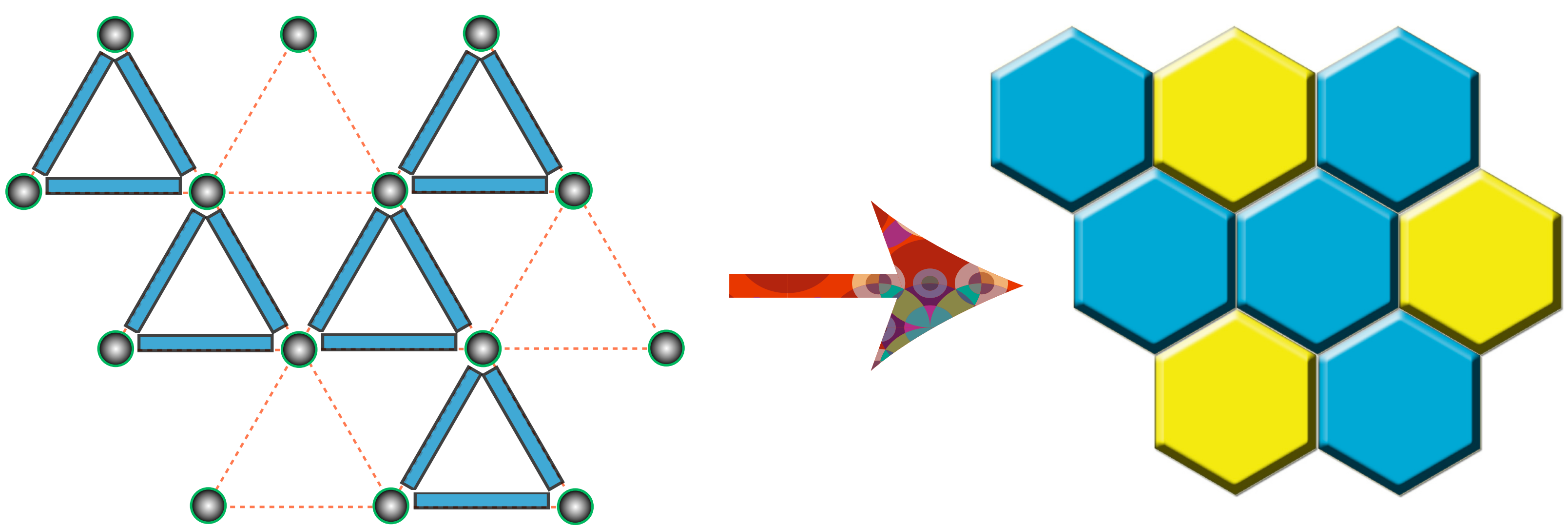


(2) On each up–pointing triangle – 8 configurations – may as well reduce vis–á–vis connectivity properties:



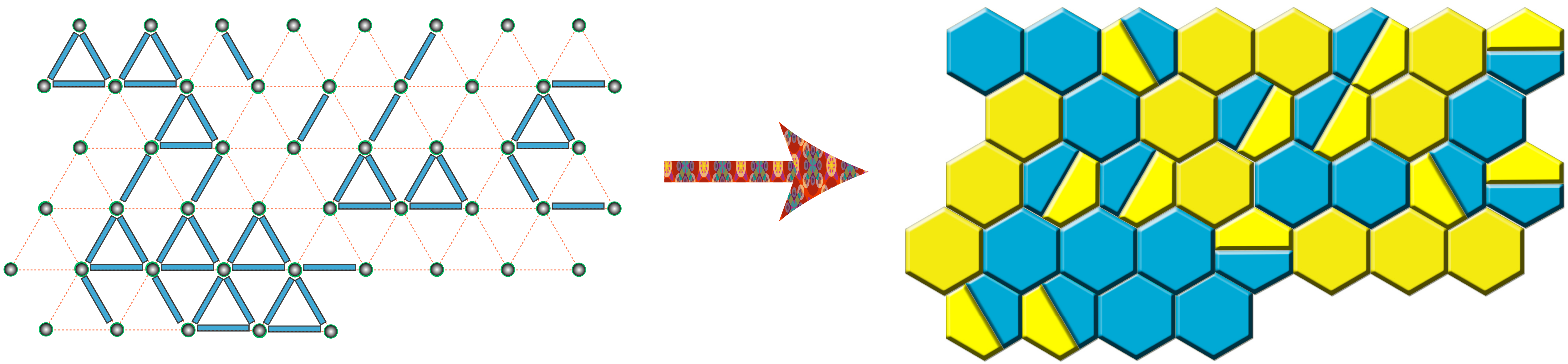
(3) Now a locally correlated percolation problem. Self–dual (via ★–▲ transformation) at $a = e$. And critical – $ae > 2s^2$ [CL].

(4) Note, $s = 0$ (i.e. $a+e = 1$) is exactly triangular site percolation problem:



Claim:

Add in single bond events (probability $s \neq 0$) $\Longleftrightarrow$ Introducing split hexagons into the problem.

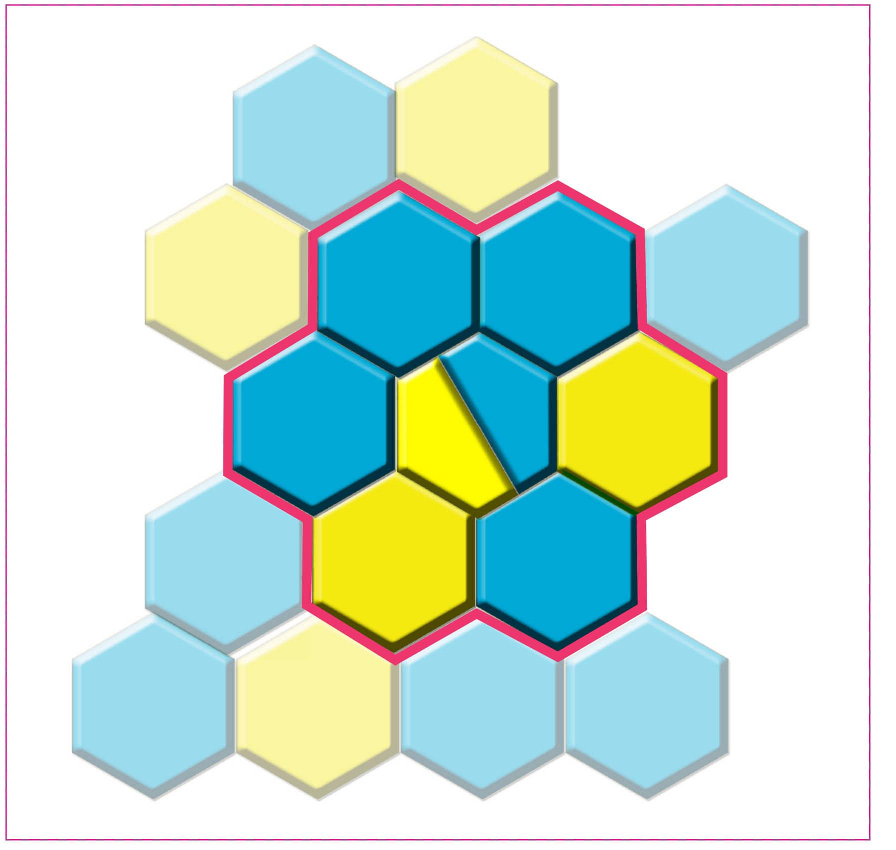


Remark:

Only three out of six possible splits of hexagons are present, hence no color switching symmetry. But each mixed hexagon has reflection symmetry through the *y*-axis and reflection through the *x*-axis followed by color reverse symmetry.

Unfortunately, full triangular bond lattice problem too hard. Need (local) correlations.

Geometric Setup



Objects of consideration:

flowers, *irises*, *petals*.

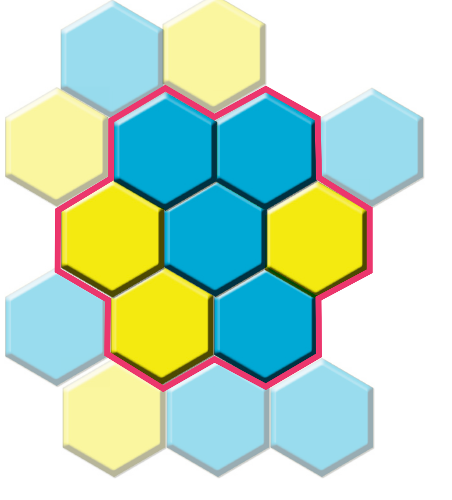
Tile the domain with hexagons, some of which are designated to be irises, such that flowers are disjoint.

Rules

- Non-irises: blue or yellow, with prob 1/2.

- Iris: blue, yellow, or mixed, with prob a, a ($a \equiv e$) and s.
  (so $2a+3s = 1$)

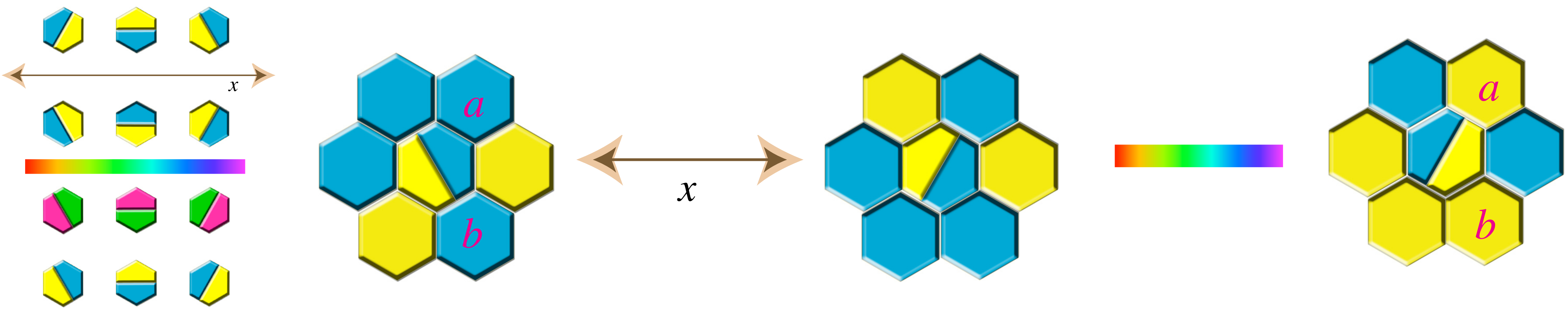
EXCEPT



- In *triggering* situations, where the iris ceases to be an iris. Note this introduces local correlations.

- Disjoint flowers are independent.

Hope to restore some color symmetry flower by flower. Indication this may work:



Reflection/color reversal gives 1-1 and onto map between the colors.

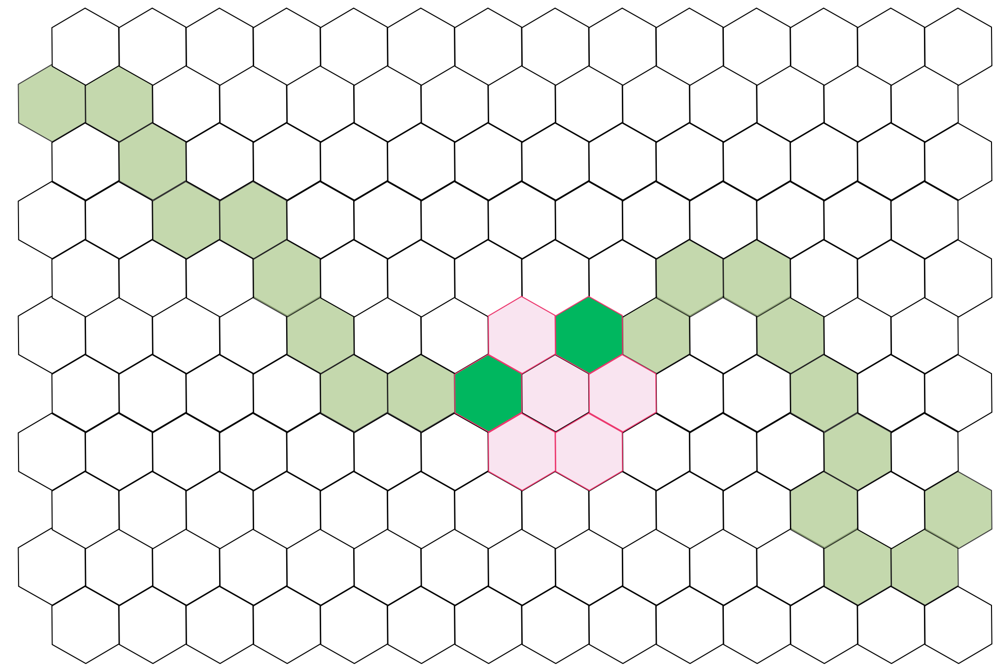
Not good enough. Need triggering.

Triggering

- $\frac{3}{16}$ of all possible configurations on a flower.
- The $$price$$ we pay:
    - Lose FKG in general (but still have it for path events)
    - A host of other difficulties to follow.
- On the bright side, these deviations due to triggering reassure us that our model is indeed different from the triangular site model and cannot be viewed as an "easy" limit of it.

We have no microscopic color symmetry, so need to consider paths "modulo flowers".

Path Designates



A path going through a flower enters at some *entrance petal* and exits at some *exit petal*.

A *path designate* specifies the path outside of flowers but only specifies the entrance/exit petals for flowers - in order.

Given a path designate $\mathscr{P}$, we let $\mathscr{P}_B$ denote the event that there is a *realization* of $\mathscr{P}$ in blue. Similar for $\mathscr{P}_Y$.

We generalize these notions in the obvious way to the case of multiple flowers and multiple visits to a single flower.

As collections of paths, not useful as a partition of the configuration space.

As geometric objects, problematic since not specific enough.

BUT ESSENTIAL FOR OBTAINING COLOR SYMMETRY.

For our purposes, we do not let a path designate start on an iris.

We are now ready to state a basic result (simplest of its type):

Lemma 1

Let $\mathbf{r}$ and $\mathbf{r}'$ denote (non–iris) hexagons. Let $K^B_{\mathbf{rr}'}$ denote the event of a blue path between $\mathbf{r}$ and $\mathbf{r}'$, and similarly for $K^Y_{\mathbf{rr}'}$. Let $\kappa^B_{\mathbf{rr}'} = \mathbb{P}(K^B_{\mathbf{rr}'})$, with a similar definition for $\kappa^Y_{\mathbf{rr}'}$. Then

$$\kappa^B_{\mathbf{rr}'} = \kappa^Y_{\mathbf{rr}'}.$$

We will prove the result flower by flower and then concatenate.

Let $\mathscr{F}$ denote a flower and let $\mathscr{D}$ denote a collection of petals of $\mathscr{F}$. Let $T^B_{\mathscr{D}}$ denote the event that all the petals in $\mathscr{D}$ are blue and that they are blue connected in the flower. Let $T^Y_{\mathscr{D}}$ denote a similar event in yellow.

We have the following result on local color symmetry:

Lemma 1.1 For all $\mathscr{D}$,

$$\mathbb{P}(T^B_{\mathscr{D}}) = \mathbb{P}(T^Y_{\mathscr{D}}).$$

We will in fact need the multiset version of Lemma 1.1 (i.e. $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$ and $T^Y_{\mathscr{D}_1 \ldots \mathscr{D}_k}$) but due to limitations of flower size, these cases do not present any additional difficulty.

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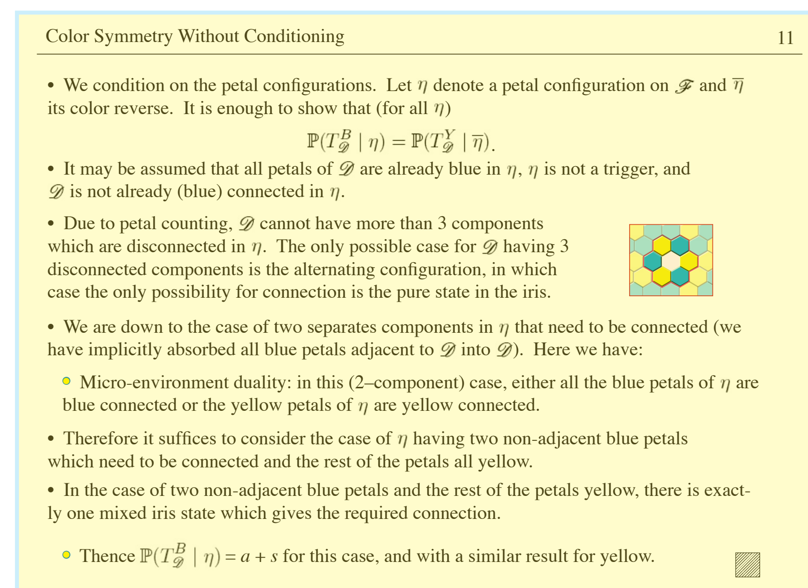


Color Symmetry Without Conditioning 11

- We condition on the petal configurations. Let η denote a petal configuration on $\mathscr{F}$ and $\overline{\eta}$ its color reverse. It is enough to show that (for all η)

$$\mathbb{P}(T^B_{\mathscr{D}} \mid \eta) = \mathbb{P}(T^Y_{\mathscr{D}} \mid \overline{\eta}).$$

- It may be assumed that all petals of $\mathscr{D}$ are already blue in η, η is not a trigger, and $\mathscr{D}$ is not already (blue) connected in η.
- Due to petal counting, $\mathscr{D}$ cannot have more than 3 components which are disconnected in η. The only possible case for $\mathscr{D}$ having 3 disconnected components is the alternating configuration, in which case the only possibility for connection is the pure state in the iris.
- We are down to the case of two separates components in η that need to be connected (we have implicitly absorbed all blue petals adjacent to $\mathscr{D}$ into $\mathscr{D}$). Here we have:
  - Micro-environment duality: in this (2–component) case, either all the blue petals of η are blue connected or the yellow petals of η are yellow connected.
- Therefore it suffices to consider the case of η having two non-adjacent blue petals which need to be connected and the rest of the petals all yellow.
- In the case of two non-adjacent blue petals and the rest of the petals yellow, there is exactly one mixed iris state which gives the required connection.
  - Thence $\mathbb{P}(T^B_{\mathscr{D}} \mid \eta) = a + s$ for this case, and with a similar result for yellow.

We will in fact need the multiset version of Lemma 1.1 (i.e. $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$ and $T^Y_{\mathscr{D}_1 \ldots \mathscr{D}_k}$) but due to limitations of flower size, these cases do not present any additional difficulty.

Given Lemma 1.1, the proof of Lemma 1 is almost immediate.

- First observe that if we let $\Pi_{\mathbf{rr}'}$ denote the collection of path designates starting at $\mathbf{r}$ and ending at $\mathbf{r}'$, then

$$\kappa^B_{\mathbf{rr}'} = \mathbb{P}(\bigcup_{\mathscr{P}\in\Pi_{\mathbf{rr}'}} \mathscr{P}_B)$$

and similarly for $\kappa^Y_{\mathbf{rr}'}$.

- Since the union is not disjoint, we will use inclusion-exclusion and prove equality on a term by term basis (note that $|\Pi_{\mathbf{rr}'}| < \infty$).

- We need some notation: if $\mathscr{P}$ is a path designate, we write

$$\mathscr{P} = [H_{\mathbf{r}1}, (\mathscr{F}_1, h^e_1, h^x_1), H_{12}, (\mathscr{F}_2, h^e_2, h^x_2), H_{23}, \dots, (\mathscr{F}_K, h^e_K, h^x_K), H_{K\mathbf{r}'}],$$

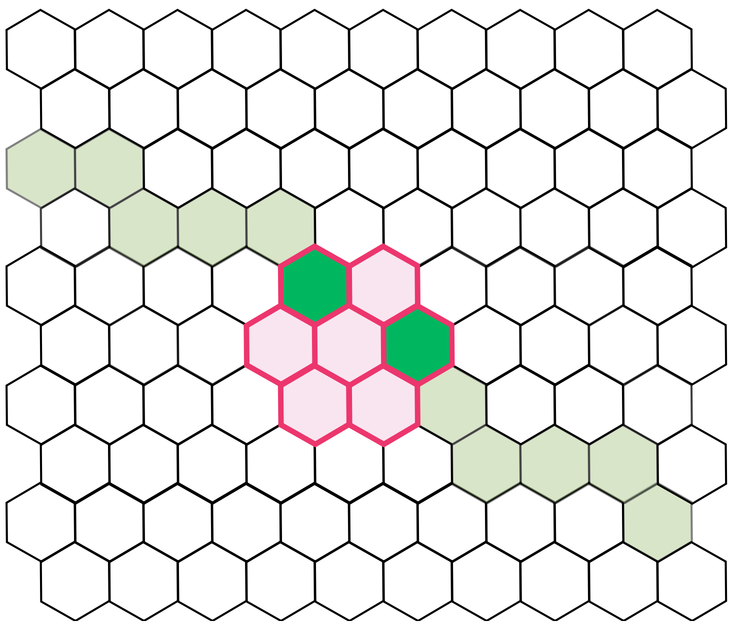
where $\mathscr{F}_1, \dots, \mathscr{F}_K$ are flowers, h^e_j and h^x_j are entrance and exit petals, $H_{j,j+1}$ is a path in the complement of flowers connecting h^x_j to h^e_{j+1}, and $\mathbf{r}$ is used to denote the hexagon at $\mathbf{r}$, etc.

- Assume for simplicity each flower is used only once. Then we have

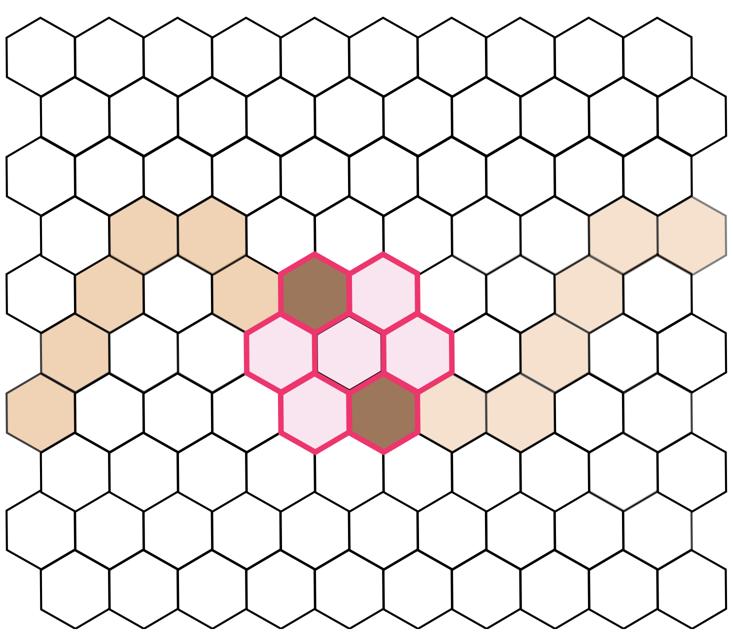
$$\mathbb{P}(\mathscr{P}_B) = \left(\frac{1}{2}\right)^{|H_{\mathbf{r},1}|} \mathbb{P}(T^B_{\{h^e_1,h^x_1\}}) \left(\frac{1}{2}\right)^{|H_{1,2}|} \dots \ \mathbb{P}(T^B_{\{h^e_k,h^x_k\}}) \left(\frac{1}{2}\right)^{|H_{K,\mathbf{r}'}|}.$$

This is exactly equal to $\mathbb{P}(\mathscr{P}_Y)$ by Lemma 1.1.

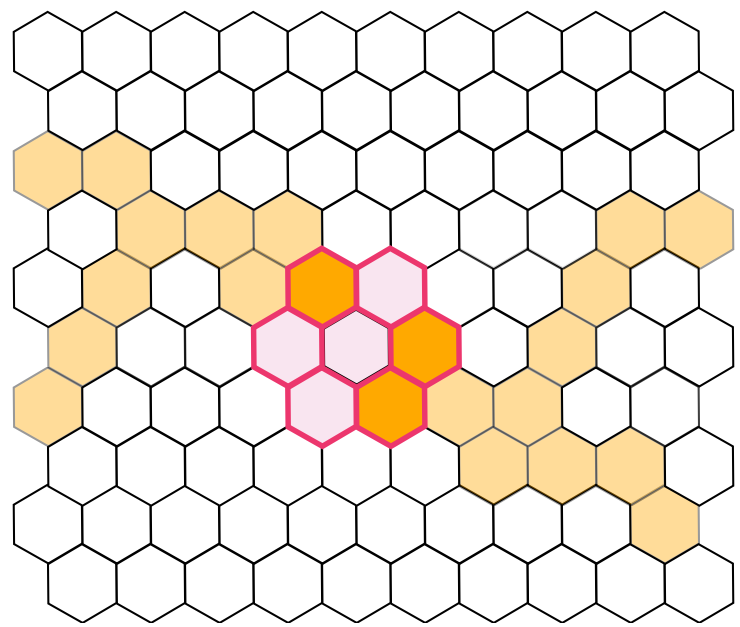
For the more general case, note that multiple paths involving the same flower (or multiple visits to the same flower) have to be treated in one piece, i.e. we will need to consider multiple $\mathscr{D}$ sets. But this is exactly the content of Lemma 1.1; we are done.



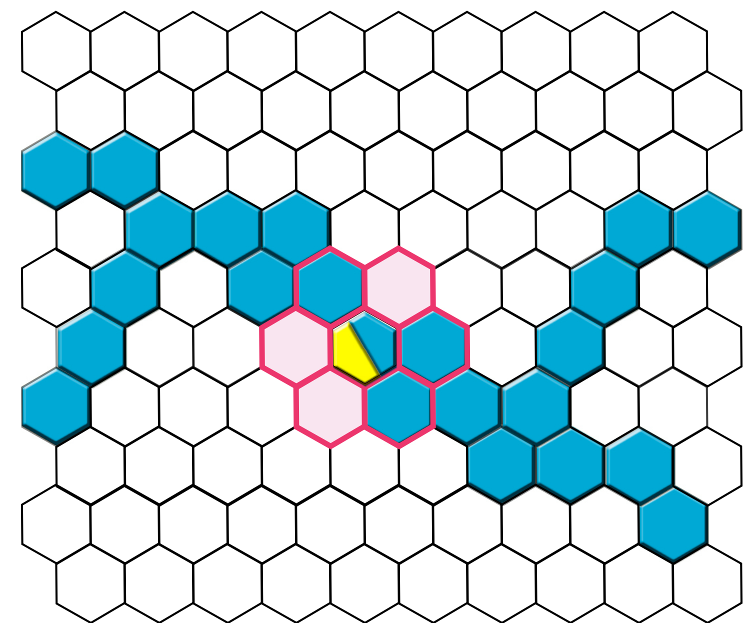
$\mathscr{P}_1, \mathscr{D}_1$



$\mathscr{P}_2, \mathscr{D}_2$



$\mathscr{P}_1 \cup \mathscr{P}_2 \quad \mathscr{D}_1, \mathscr{D}_2$



$(\mathscr{P}_1)_B \cap (\mathscr{P}_2)_B$

Lemma 1 + periodic floral arrangement + $ae \geq 2s^2$ can be used to establish typical critical behavior:

- No percolation of yellow or blue.

- Rings in annuli (with uniform probability) @ all scales.

- Power law bounds on connectivities.

But for us, this is just the beginning. We must face up to problem of color symmetry for transmissions in presence of *conditioned paths*.

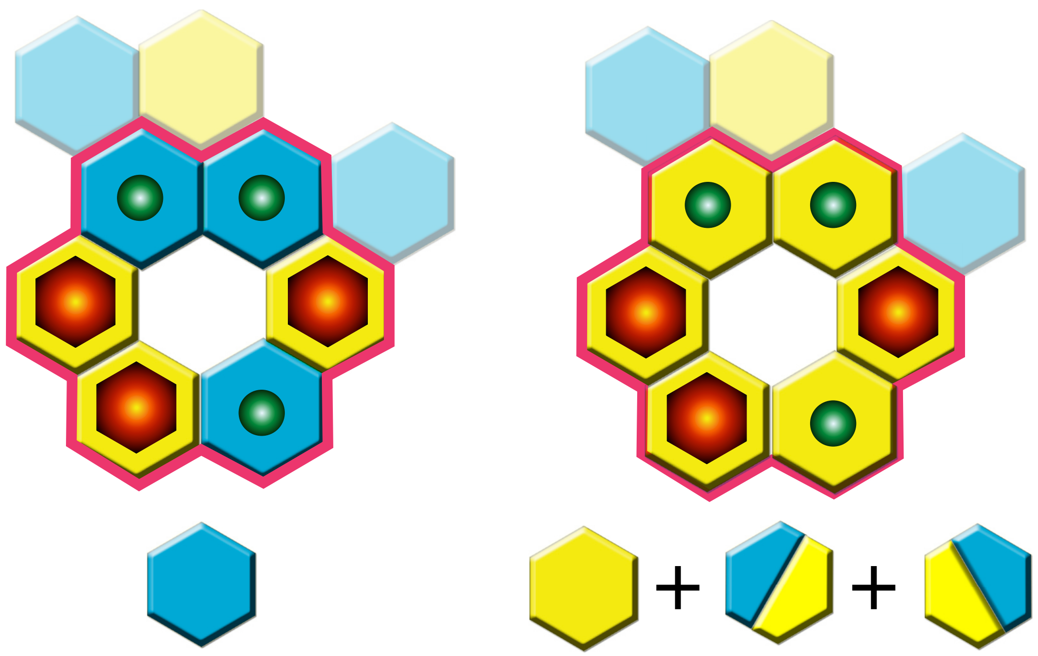
For CR need to change color in presence of *conditioning*.

PROBLEM

Example:

Conditioned Sites

Transmission Ports



$\frac{1}{2}$ (Trigger) $a + 2s$

SOLUTION

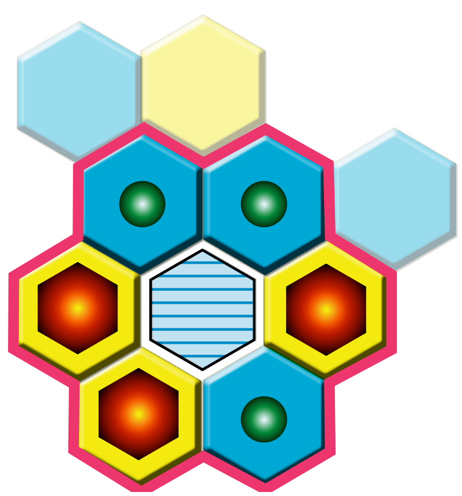
Rethink the meaning of disjoint

DEUS EX MACHINA

When blue at disadvantage, allow blue conditioned petals to be shared with some probability.

When blue at advantage, forbid from touching blue petals used by the conditioned set.

PREVIOUS EXAMPLE



Always fine

with probability

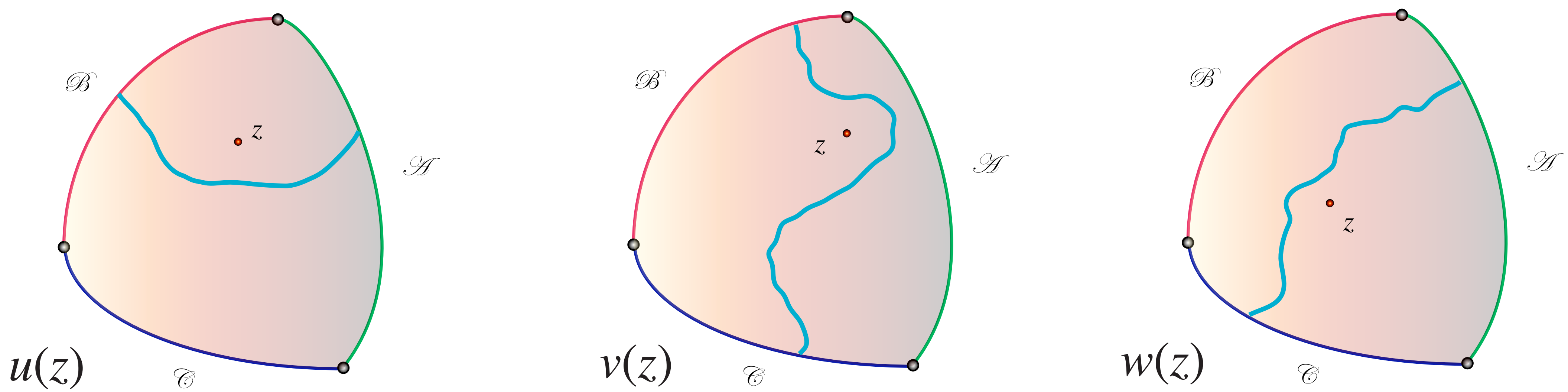
$$\frac{s}{2(a+2s)}.$$

Lemma 2

There exists a set of $*$-rules (laws for all relevant random variables) such that for any points x and y the probability of a $*$-transmission from x to y in the "complement" of any paths Γ_b & Γ_y is the same for blue as it is for yellow.

Here $*$-transmission means, depending on the values of the auxiliary random variables and the relevant colors involved, the possibility of leeway provided for the sharing of hexagons and/or adherence to no close encounter rules.

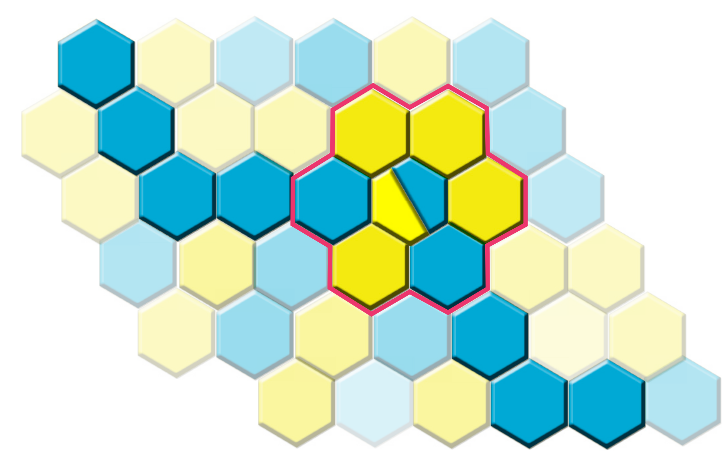
What does all this mean for our functions u_N, v_N and w_N? (N denotes lattice spacing of N^{-1})



STRATEGY

I. Prove what we want for $*$-versions of the functions: u_N^*, v_N^* and w_N^*.

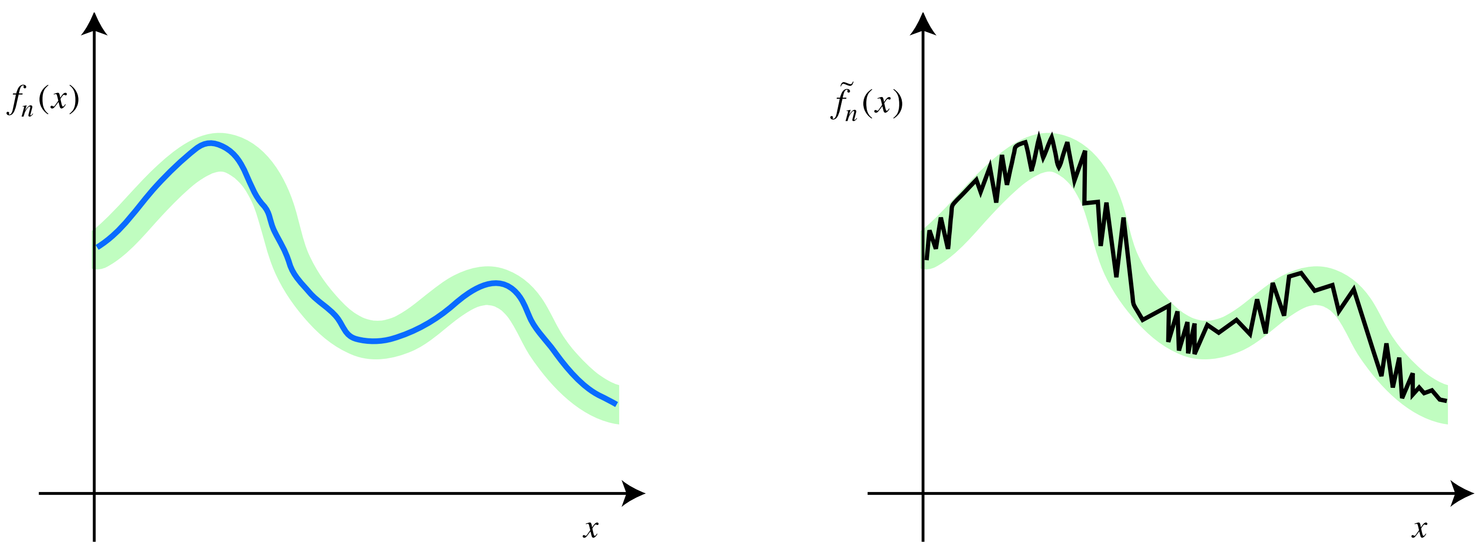
II. Then do some analysis to show e.g. $|u_N - u_N^*| \to 0$.



$u_N^* \notin \{0,1\}$

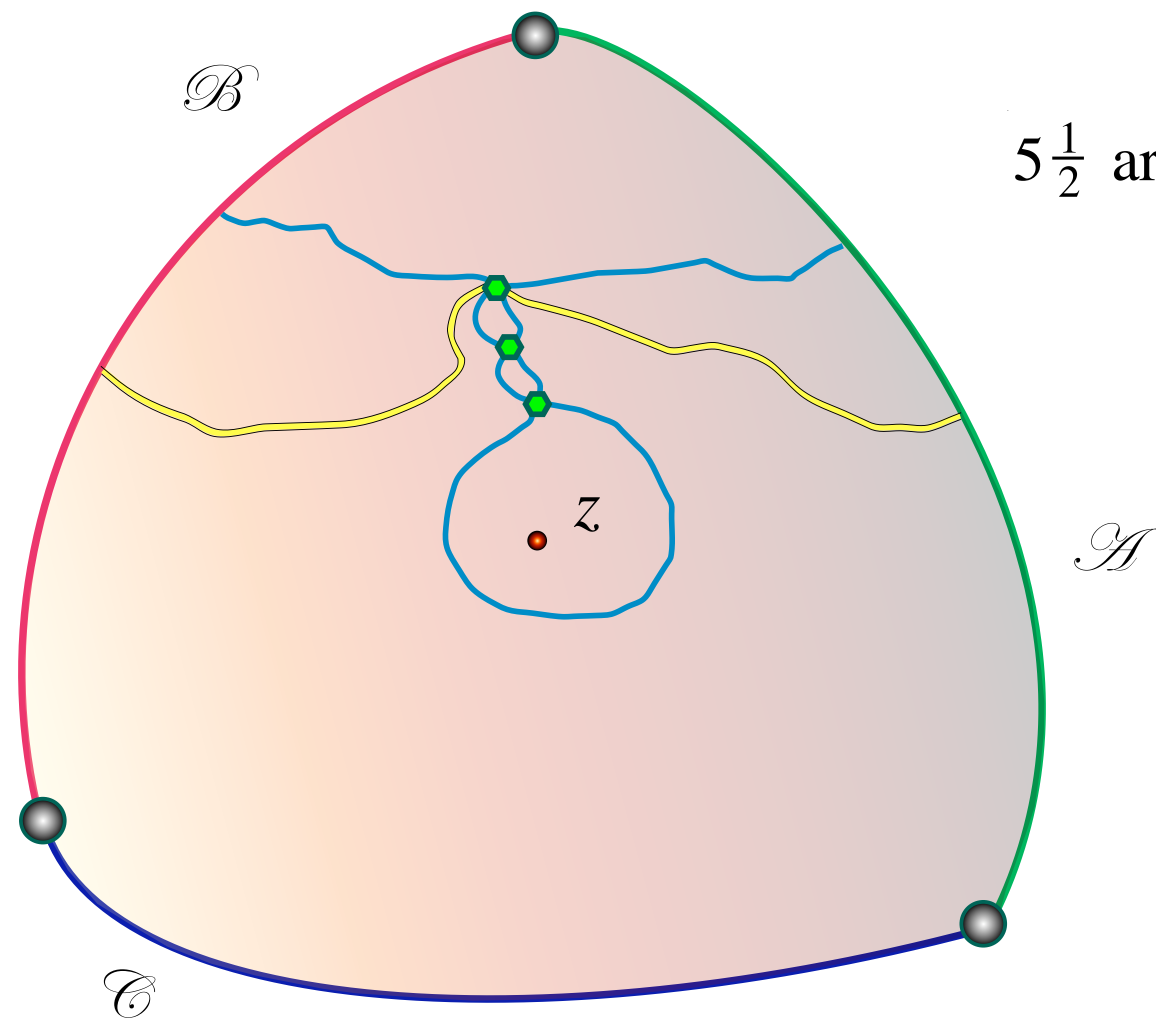
color switching lemma + a contour argument + more gives (I).

General picture of (II) is



- Path satisfying "event" only come near z with vanishingly small probability.

- Path of a configuration in $\mathcal{U}^*_N(z)\,\Delta\,\mathcal{U}_N(z)$ not close to z lead to "five and a half" arms, which occur with vanishingly small probability.



$5\frac{1}{2}$ arm event

The $$price$$ of color symmetry:

I. FKG inequality and RSW lemmas.

- FKG was ostensibly difficult, but the assumption of $a^2 \geq 2s^2$ and the result in [CL] made it easy.
- For RSW, among other difficulties, had to actually read Kesten's book.
- Tragedy of RSW: lost rights to arbitrary floral arrangements.

II. Arms and Exponents.

- A five and a half arm argument, along with a three arm argument in the complement of a line segment was needed to show equivalence of Carleson-Cardy functions.
- Due to local correlations, standard KvB or Reimer's inequality does not apply, needed old fashioned conditioning argument.

III. Full Flower vs. "Used" Flower.

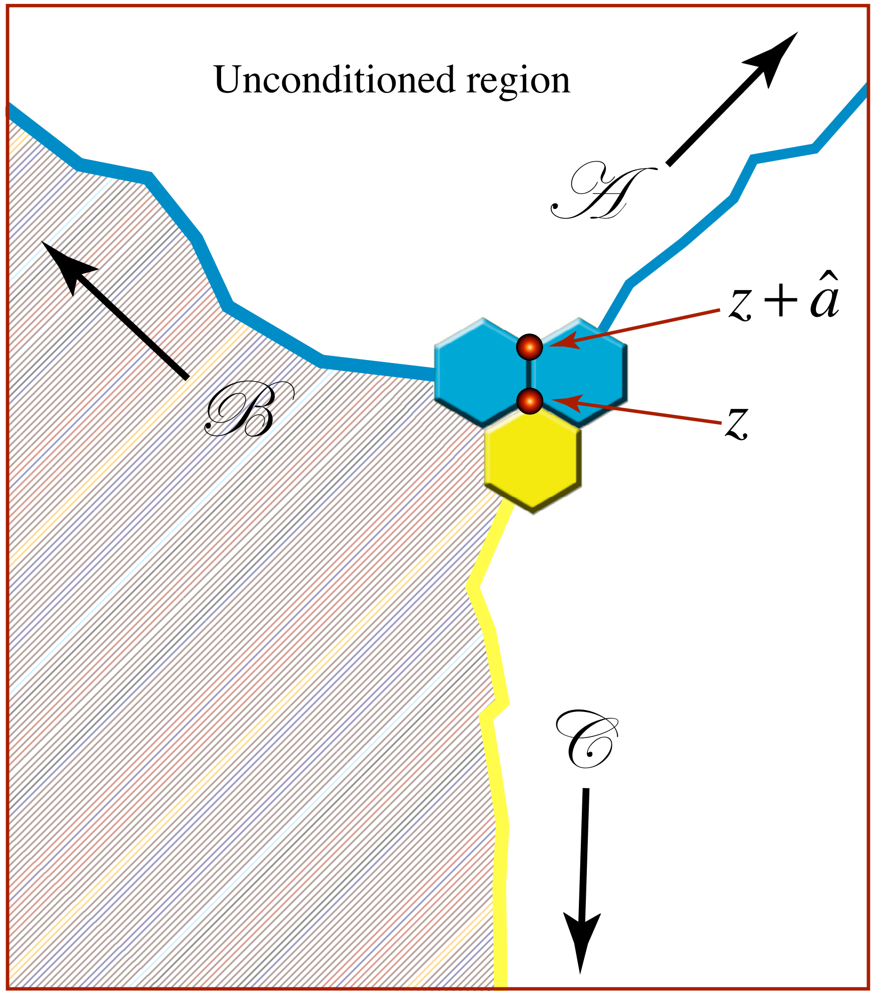
- This was needed in the conditioning argument in II.
- Seemingly "obvious", but involved meticulous and systematic consideration of all possibilities.

IV. The Iris in Cauchy-Riemann Switch.

- No sensible mechanism to have path designate start @ iris. CR–relations require effort.

V. Producing the Lowest Path for Conditioning (loop erasure).

Had to condition on “lowest” paths, need to ensure some paths are self-avoiding/non-self-touching.



PROBLEM

The $*$-paths are NOT self-avoiding/non-self-touching.

QUICK CURE

Take a geometric path and delete all loops.

COMPLICATIONS

I. Must keep loops that “capture” z.

II. Random variables may cause unwanted “dumping” after deletion of loops.

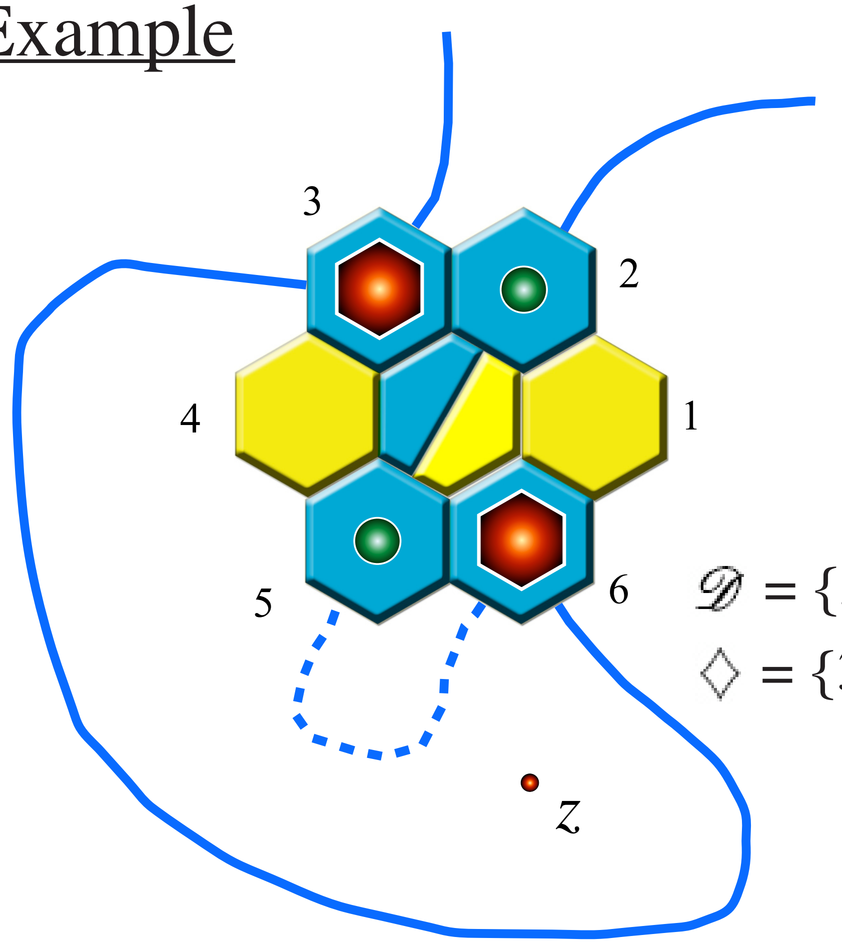
When is a path $*$-good?

- First pass through flower is “free”.
- Pass N through flower takes petals used by first N-1 passes as conditioned set.

Must receive “permission” from all relevant random variables!

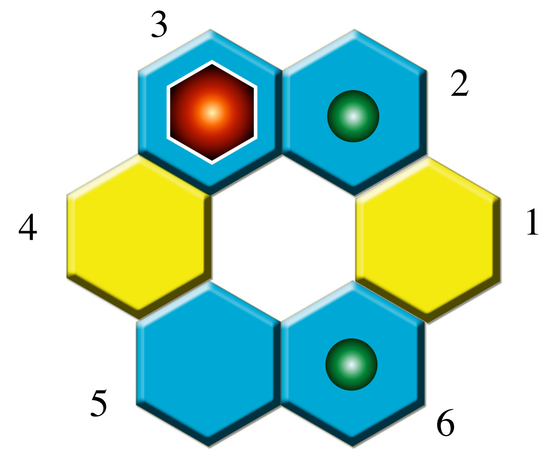
In fact, the statement that the fully reduced version of Γ (i.e. all loops erased except for the one necessary for "capture" of z) satisfies the event is false:

<u>Example</u>



$\mathscr{D} = \{5,2\}$

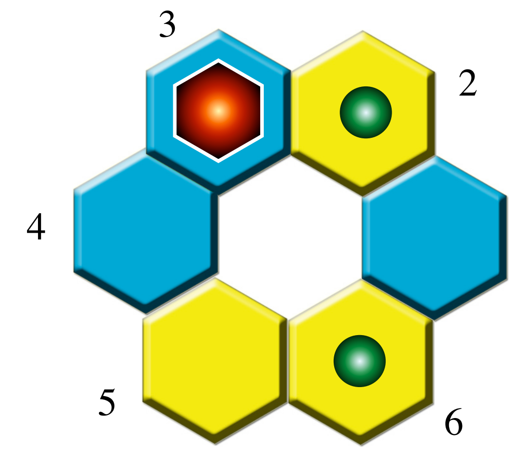
$\diamondsuit = \{3,6\}$



$\mathscr{D} = \{2,6\}$

$\diamondsuit = \{3\}$

Transmission: $(a + 2s)$



<u>Trigger</u>

Transmission: $\frac{1}{2}$

THE TRUTH

Can reduce half the path (from boundary to first bottleneck of loop with z in interior).

This is all we need.

(I) Wrap–Up. After much work, result is that lattice functions u_N, v_N & w_N for *this* model converge to the "Cardy–Carleson" functions.

I.e. the same result as for triangle site lattice model.

Pretty much a complete proof that the continuum limits of both systems are exactly the same; Reasonable and fairly robust statement of *universality*.

Central dogma for theory of critical phenomona since the 1960's.

(II) Limitations

(a) Not a standard (well known) percolation model.

(b) Within context of model, didn't get most complete result.

Although model does indeed have parameters –"some generality".

(c) Aside from some practical (and technical) considerations, did *not* learn much about the nature of and convergence to continuum limit – beyond what was already known.