UCLA

Department of Mathematics

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Cardy's Formula for Certain Models of the Bond Triangular Type

Joint work with L. Chayes 

Talk Outline

I Background & Smirnov's Proof

II Triangular Bond Model

III Model Under Consideration

IV Path Designates

V Color Symmetry Without Conditioning

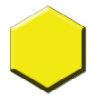
VI Color Symmetry Under Conditioning

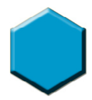
VII Crossing Probabilities

VIII Summary of Technical Difficulties

IX Loop Erasure

Work of Smirnov takes place on the triangular site lattice, equivalently hexagon tiling of $\mathbb{C}$.

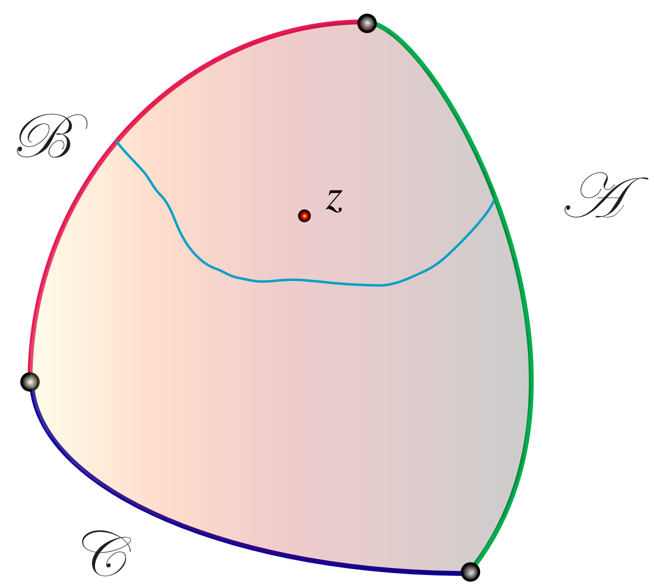
Hexagon yellow:  probability p.

Hexagon blue:  probability $(1-p)$.

Critical point at $p_c = \frac{1}{2}$, i.e.

$p > p_c$	percolation of yellows
$p < p_c$	percolation of blues

<u>Central Practical Goal</u>

U(z): 

$$u(z) = \mathrm{P}\,(\mathbf{U}(z)),$$

then as lattice spacing tends to zero, $u(z)$ converges to an "appropriate" harmonic functions. Similarly define the functions v and w.

- With boundary and analyticity conditions (more later), u and related functions are uniquely specified (conformally invariant).
- The said harmonic functions are linear on the equilateral triangle and satisfy Cardy's Formula.

Key Ideas

I. Harmonic Triples

- $u + \frac{i}{\sqrt{3}}(v - w)$ and all cyclic permutations of this are analytic functions.
    - The above analytic functions are not independent, e.g. they add up to 1; so in actuality only have one analytic function.
- We have Cauchy-Riemann type equations of the form

$$D_{\hat{s}}u = D_{(\tau\hat{s})}v$$

and cyclic permutations thereof. Here $\hat{s}$ is any direction and $\tau = \exp(\frac{2\pi i}{3})$.

- *u, v* and *w* are conformal transformations of the linear (Carleson-Cardy) functions from the equilateral triangle.
    - Hence Cardy’s formula comes for “free” once it is shown that (*u, v, w*) converge to the appropriate harmonic triple on “any” conformal domain.

II. Lattice Functions

- Boundary conditions will come more or less directly from definition of functions.
    - These conditions are lattice independent and only uses Harris rings type arguments.
- Cauchy-Riemann Equations? Certainly not. Herein lies the main difficulty.

Discrete Derivatives and Color Switching

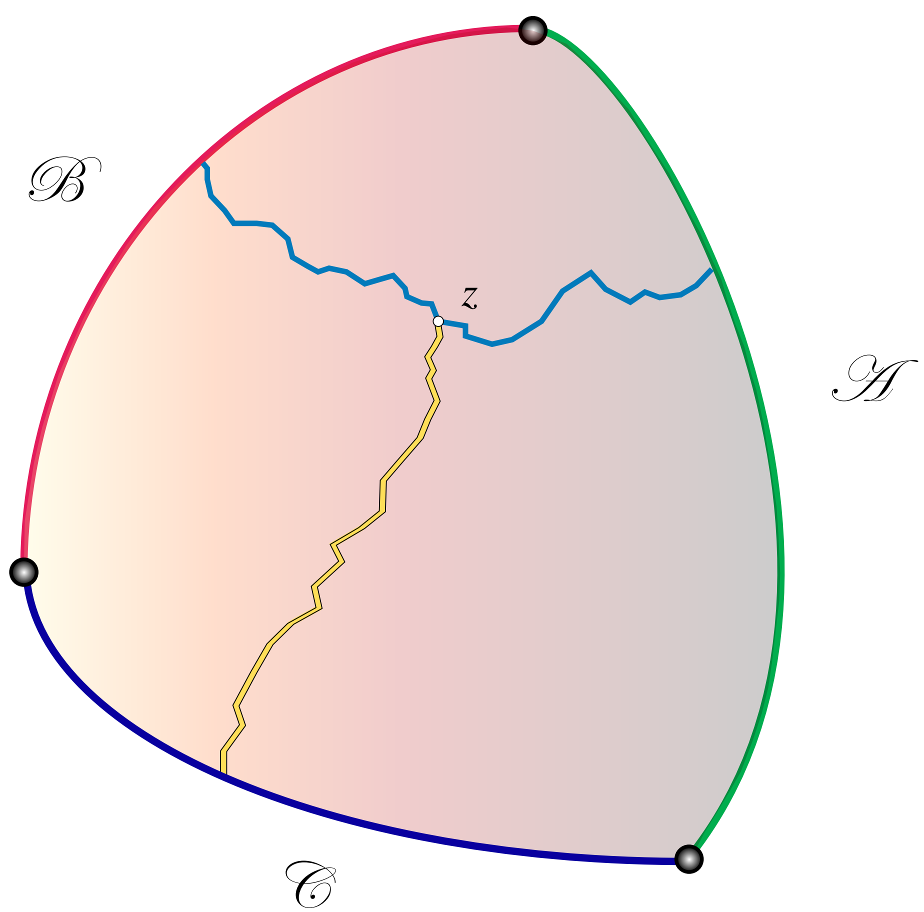
- The discrete derivative is given by

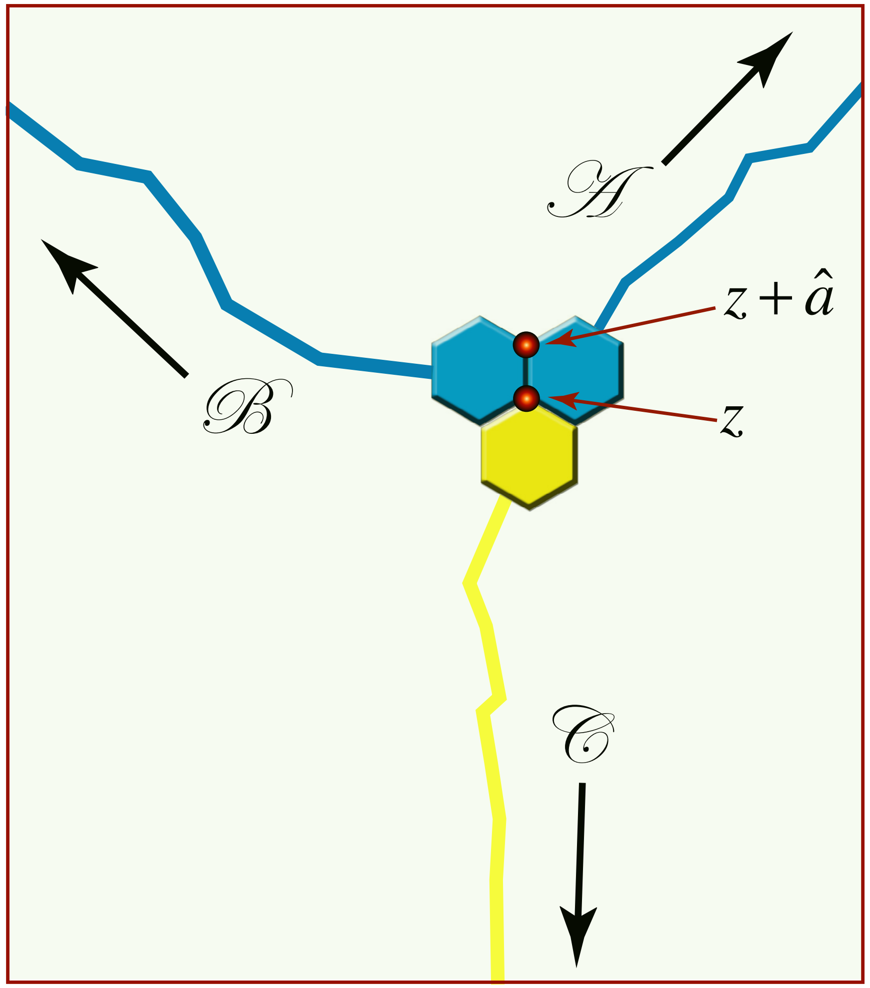
$$u(z+\hat{a})-u(z)$$

which is seen to equal

$$\mathrm{P}\,[\mathbf{U}(z+\hat{a}) \setminus \mathbf{U}(z)] - \mathrm{P}\,[\mathbf{U}(z) \setminus \mathbf{U}(z+\hat{a})].$$

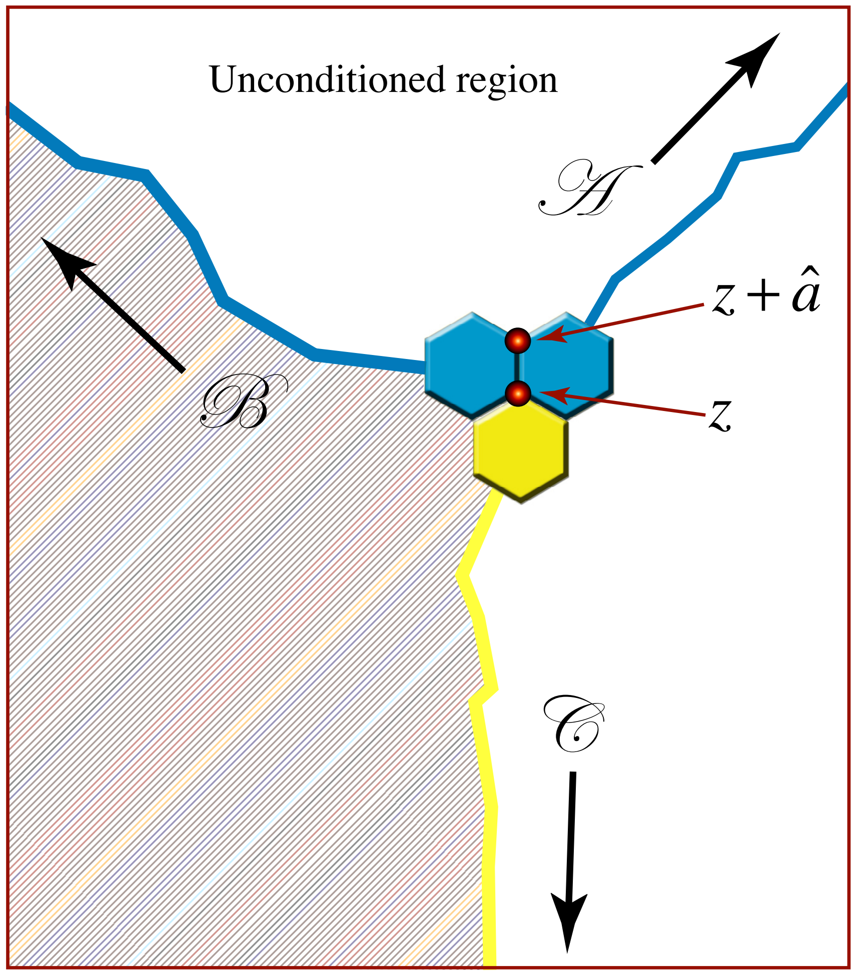
The first term in the above display we denote by $\mathbf{U}_a^+(z)$ and the second we denote by $\mathbf{U}_a^-(z)$.

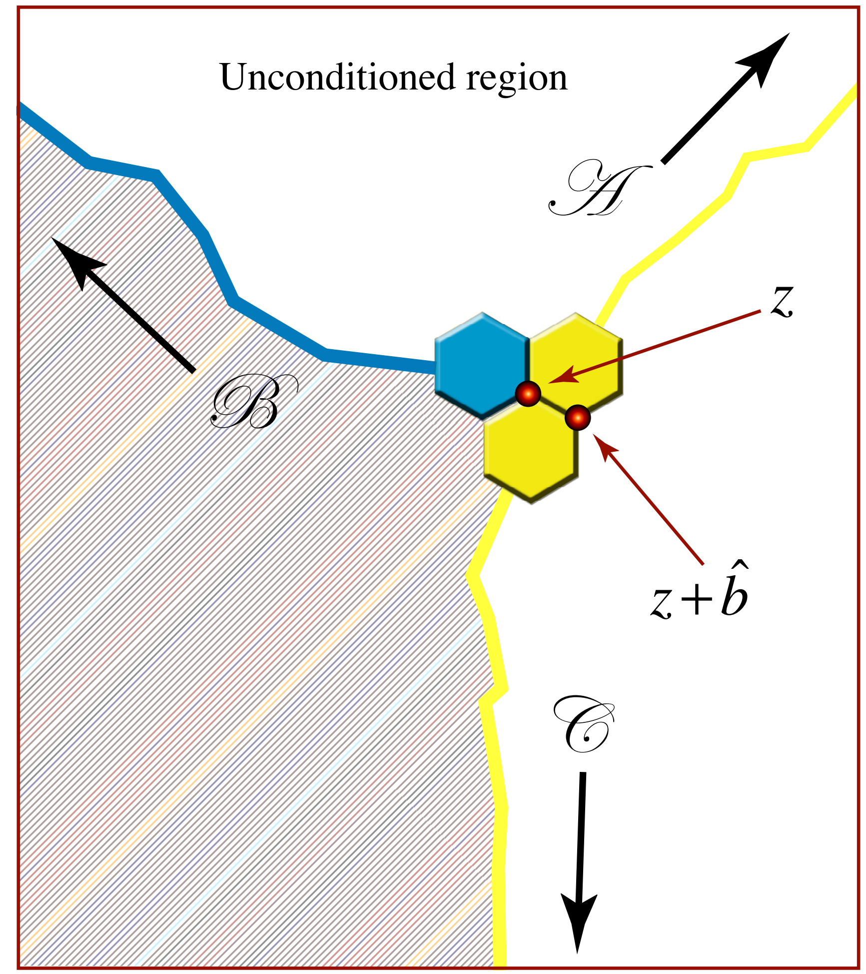




- Now we can derive the necessary CR relations: let $\hat{a}$ & $\hat{b}$ be two lattice vectors as shown, then

$$\mathbf{U}_a^+ = W_b^+$$





i.e. the probabilities of the two "CR–pieces" are the same.
(Some conditional/partitioning arguments were used here).

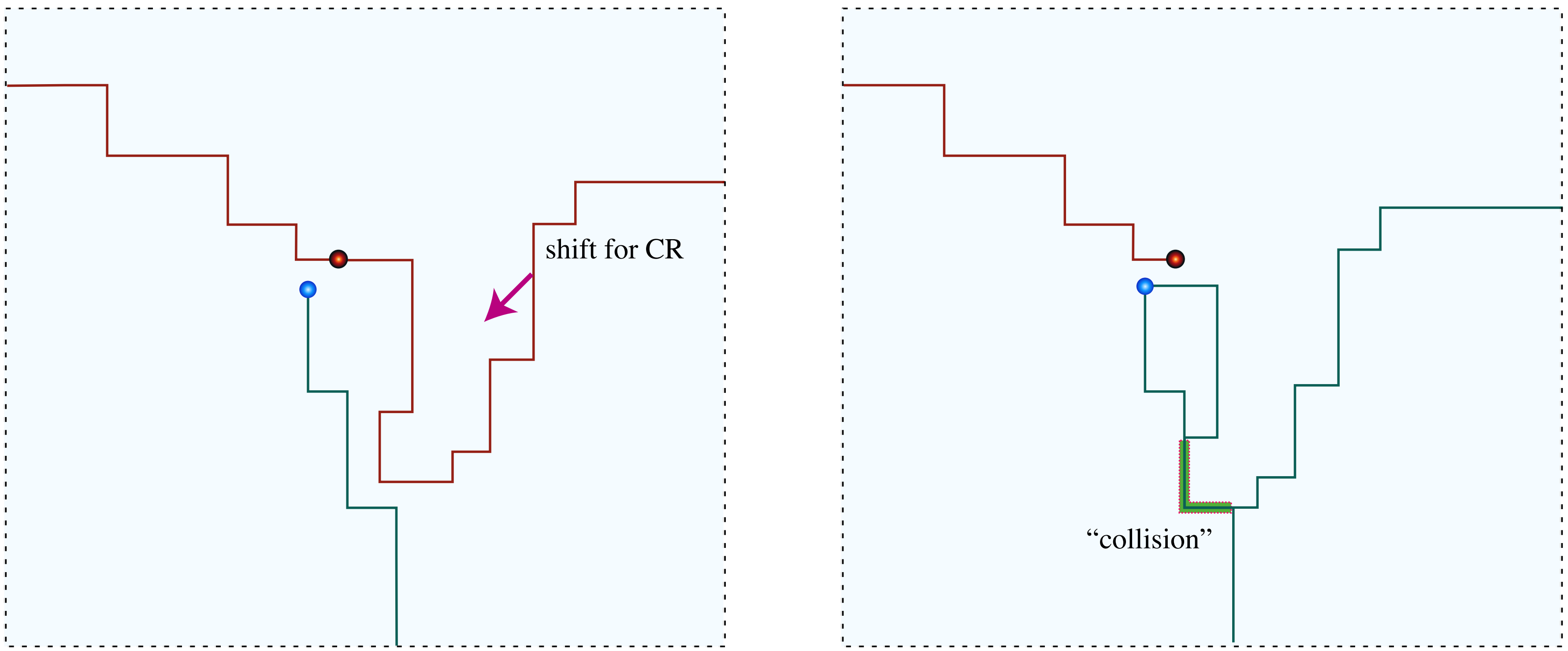
- Without going into details, these relations, together with a contour integration, is enough to push through a proof.
- It is a miracle of the triangular site lattice that these innocuous looking CR relations hold without apology.

Difficulties With Other Lattices

A number of items make the triangular site lattice special:

- On the macroscopic level, any reasonable candidate function will have derivatives (pieces) which are of three–arm type. This, together with lattice structure (not to mention linear form of Cardy's formula on an equilateral triangle) makes the triangular geometry especially well-suited for this line of attack.

- The lattice level CR–relations follow from color symmetry. Fact that $p_c = \frac{1}{2}$ certainly indicated this symmetry, however, the other percolation systems with $p_c = \frac{1}{2}$ e.g. bond square lattice, suffer from "collision" problems with any attempt to switch colors.

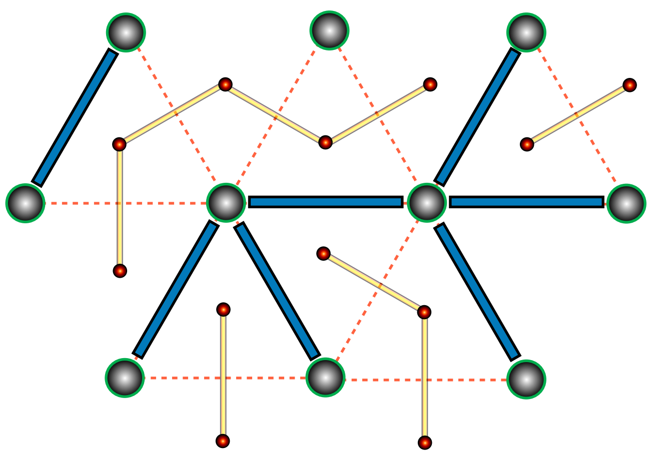


- On the other hand, for other site lattices, e.g. the square lattice, with p_c no longer $\frac{1}{2}$ (no self–dual symmetry) the connectivity properties also change when going from the direct to the dual model, which makes this approach apparently hopeless.

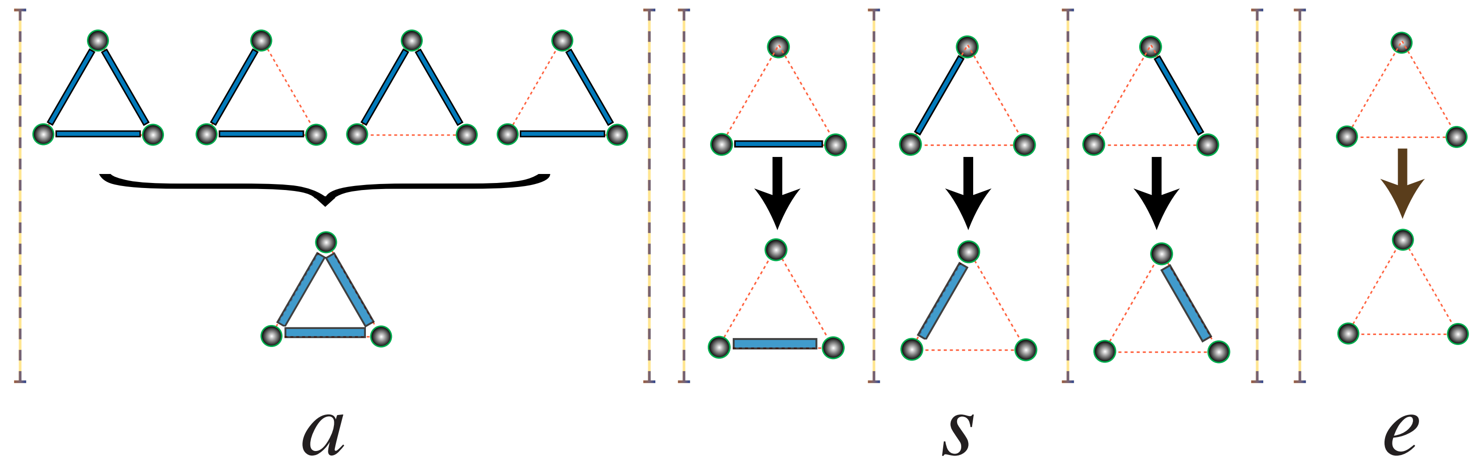
L. Chayes and H.K. Lei, *Random Cluster Models on the Triangular Lattice*, to appear in J. Statist. Phys

Model under consideration: Based on triangular lattice *bond* percolation problem.

(1) Bonds independently *blue*: p / *not–blue*: $(1–p)$.

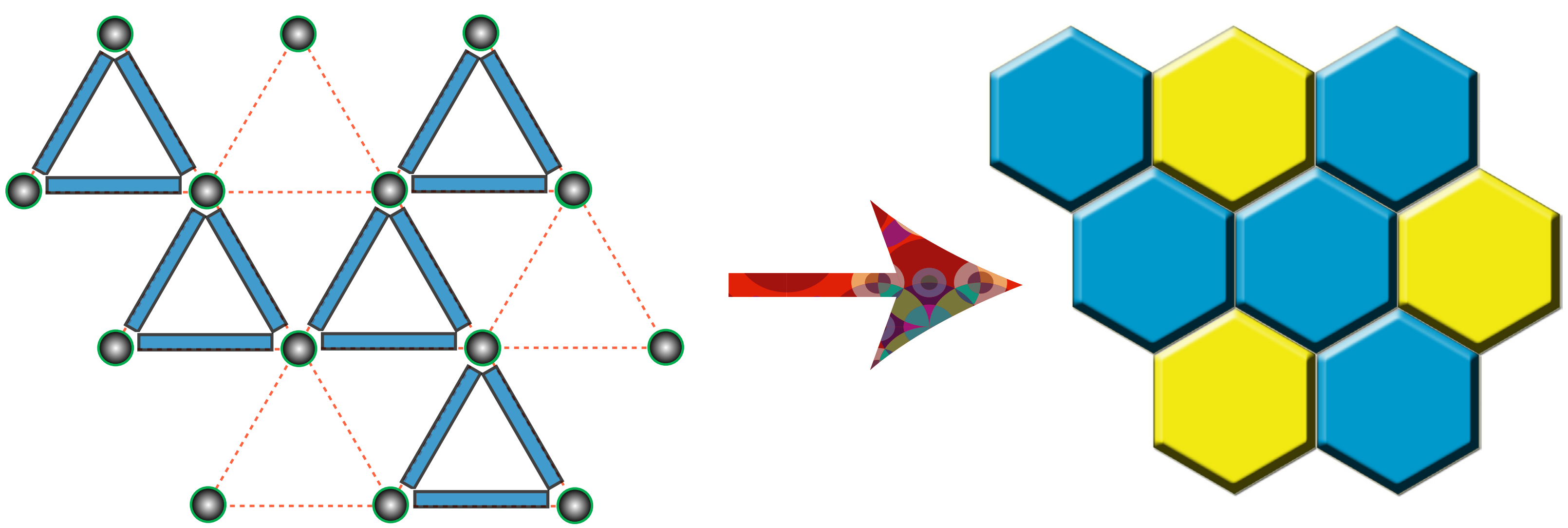


(2) On each up–pointing triangle – 8 configurations – may as well reduce vis–á–vis connectivity properties:



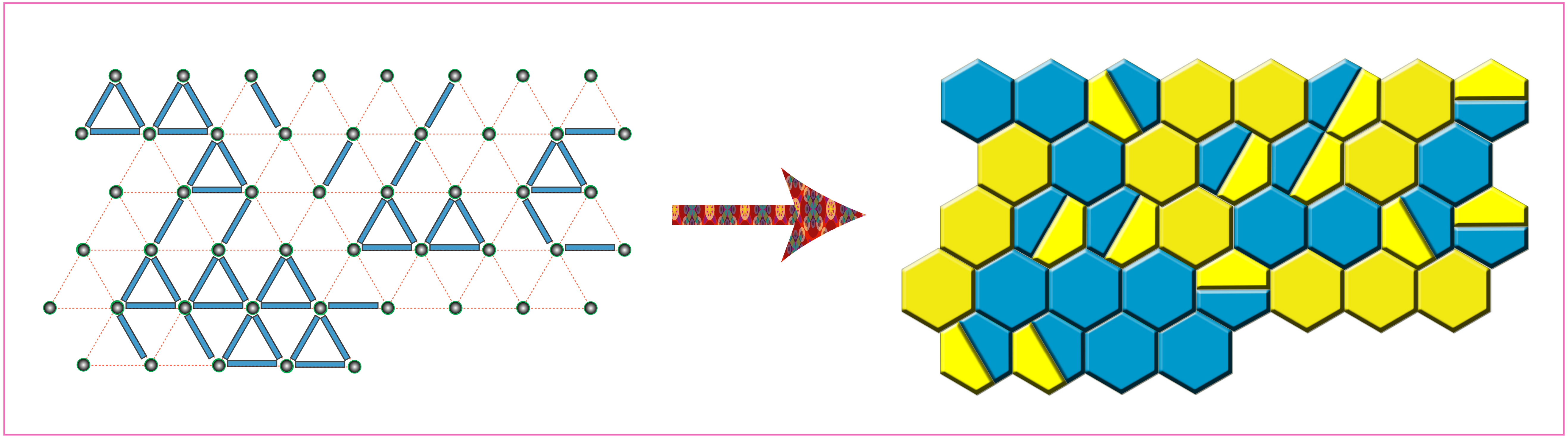
(3) Now a locally correlated percolation problem. Self–dual (via ★–▲ transformation) at $a = e$. And critical – $ae > 2s^2$ [CL].

(4) Note, $s = 0$ (i.e. $a+e = 1$) is exactly triangular site percolation problem:



<u>Claim</u>:

Add in single bond events (probability $s \neq 0$) $\Longleftrightarrow$ Introducing split hexagons into the problem.



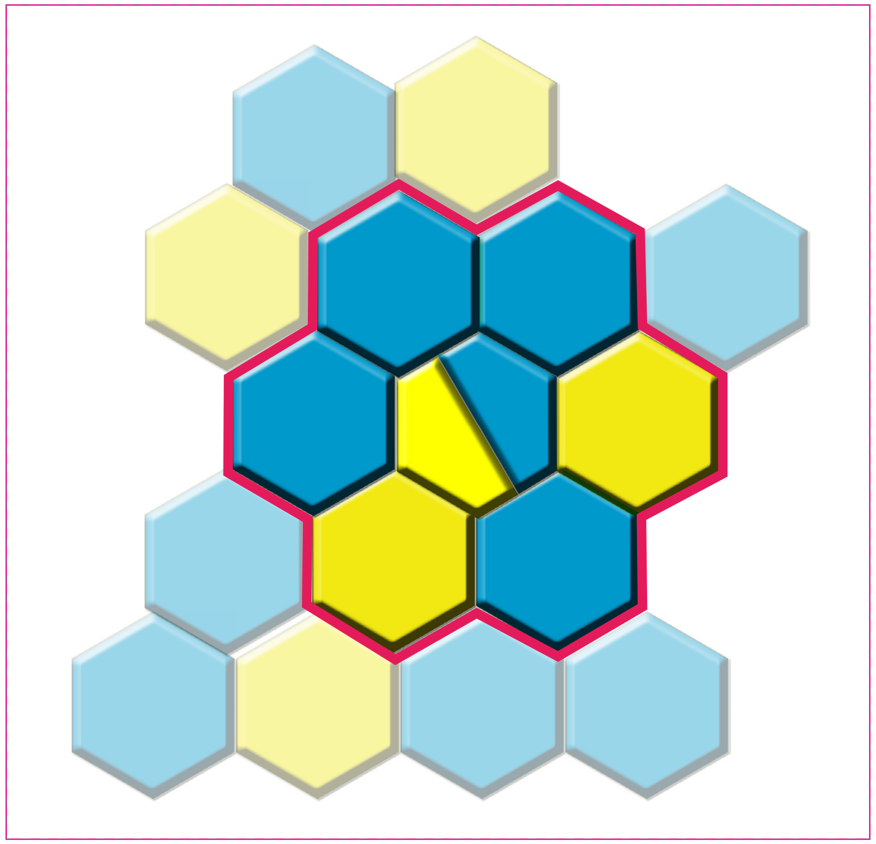
Remark:

- Only three out of six possible splits of hexagons are present, hence no color switching symmetry. But each mixed hexagon has reflection symmetry through the y-axis and reflection through the x-axis followed by color reverse symmetry.
- Now color-switching looks plausible, but must somehow allow "sharing" of hexagons – analogue of collisions.

Unfortunately, full triangular bond lattice problem too hard.

- Introduce additional (local) correlations.

Geometric Setup



- We will focus attention on a certain local arrangement of hexagons, called *flowers*.
- The hexagon in the center of a flower are called *irises*; the six hexagons surrounding an iris are called *petals*.
- We tile the domain with hexagons, where some hexagons are designated to be irises, such that:

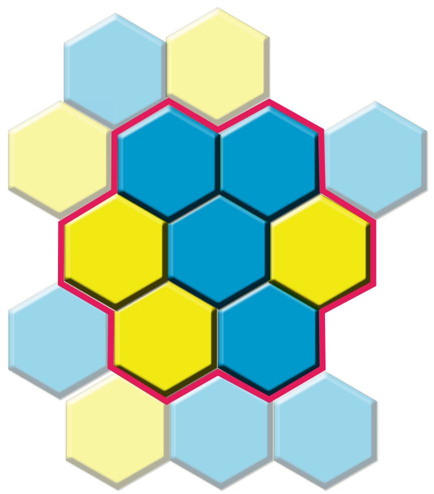
  (1) The flowers associated to the irises are disjoint.

  (2) No boundary hexagon is an iris.

Specifics

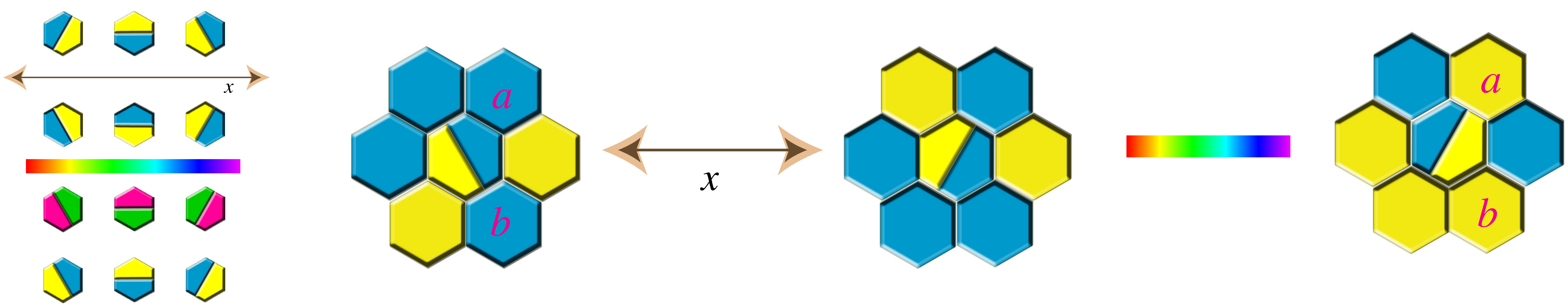
Now we specify what states each hexagon is allowed to be in.

- Petals and hexagons not part of a flower are only allowed to be pure blue or pure yellow (independently), each with probability $\frac{1}{2}$.
- Each iris is allowed to be pure blue, pure yellow, or mixed with probabilities a, a ($a \equiv e$) and s, respectively (so $2a+3s = 1$), except...
- In *triggering* situations, where the iris is only allowed to be pure blue or pure yellow, each with probability $\frac{1}{2}$. Note that this introduces local correlations in our system.
- Disjoint flowers are independent.



Flowers

- On the triangular site lattice, one has complete color symmetry at each site. We cannot hope for this on the triangular bond lattice. However, one can still hope for some form of color symmetry to be restored locally. In light of this, the flower is the smallest such local structure - with all requisite symmetries - that one can consider.
- Indeed, due to reflection plus color reversal symmetry, the probability of a connection between petal a and petal b is in fact exactly the same for yellow as it is for blue, e.g.:



- Unfortunately, one needs much more than this. It is only with the advent of *triggering* that the requisite (local) color symmetry is restored.

Triggering

- Triggering configurations consist of only $\frac{3}{16}$ of all possible configurations on a flower.
- We pay a hefty price for this restoration of local color symmetry:
  - These local correlations are enough to destroy the FKG property in general.

  (But we still have the FKG property for path events, which turns out to be sufficient for our results, in particular, that the model is critical in the usual sense of 2D percolation.)
  - A host of other difficulties to follow.
- On the bright side, these deviations due to triggering reassure us that our model is indeed different from the triangular site model and cannot be viewed as an “easy” limit of it.

As we only have color symmetry on the level of flowers, it is natural to consider collections of paths. More precisely, key to our program is the concept of

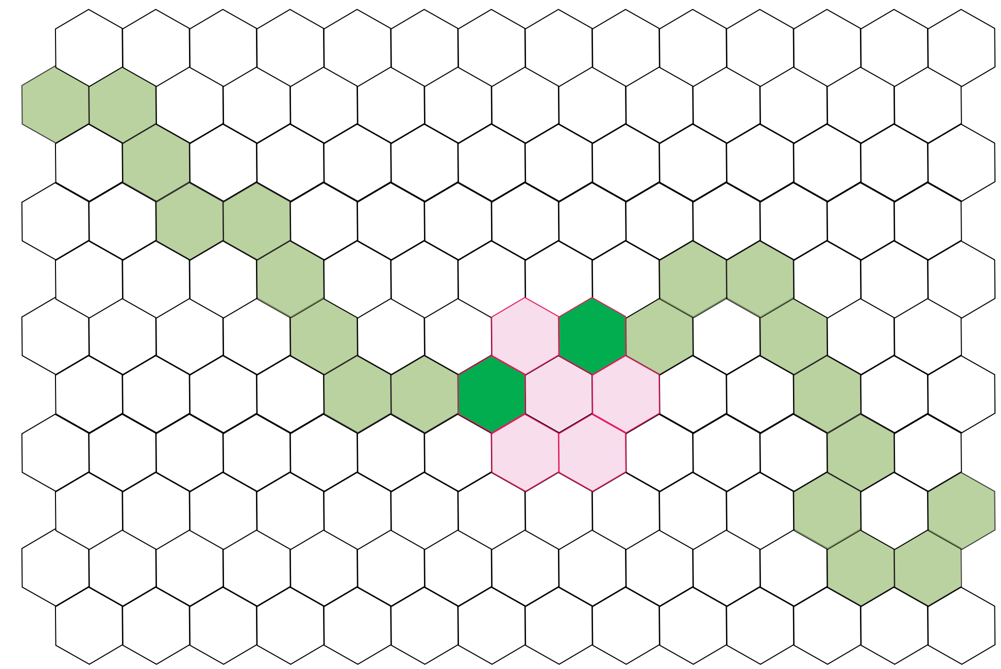
Path Designates

- Consider the case where there is only one flower in the entire domain and a path which visits this flower exactly once; then there must be an *entrance hexagon* - the very first hexagon of the flower the path uses - and an *exit hexagon* (after which the path never visits the flower again). Note that both the entrance and exit hexagons are petals.

- A *path designate* will simply specify the entrance and exit hexagons of flowers (in order) with no regard to how these two are connected within the flower; in the region complementary to the flower, the path must be specified completely.

- Given a path designate $\mathscr{P}$, we let $\mathscr{P}_B$ denote the event that there is a *realization* of $\mathscr{P}$ in blue, i.e. there is a blue path which agrees with $\mathscr{P}$ on the complement of the flower and connects the entrance and exit hexagons (both must be blue) in blue within the flower. Similar for $\mathscr{P}_Y$.

- For our purposes, we do not let a path designate start on an iris.



- We generalize these notions in the obvious way to the case of multiple flowers and multiple visits to a single flower.

- Note that a path designates can be thought of as a collections of paths, but apparently not useful as a partition of the configuration space.

We are now ready to state a basic result (simplest of its type):

Lemma 1

Let $\mathbf{r}$ and $\mathbf{r}'$ denote points (hexagons) which are not irises. Let $K^B_{\mathbf{rr}'}$ denote the event of a blue path between $\mathbf{r}$ and $\mathbf{r}'$, and similarly for $K^Y_{\mathbf{rr}'}$. Let $\kappa^B_{\mathbf{rr}'} = \mathbb{P}(K^B_{\mathbf{rr}'})$, with a similar definition for $\kappa^Y_{\mathbf{rr}'}$. Then

$$\kappa^B_{\mathbf{rr}'} = \kappa^Y_{\mathbf{rr}'}.$$

We will prove the result flower by flower and then concatenate. We make the following definition for convenience:

- Let $\mathscr{F}$ denote a flower and let $\mathscr{D}$ denote a collection of petals of $\mathscr{F}$. Let $T^B_{\mathscr{D}}$ denote the event that all the petals in $\mathscr{D}$ are blue and that they are blue connected in the flower. Let $T^Y_{\mathscr{D}}$ denote a similar event in yellow.

We have the following result on local color symmetry:

Lemma 1.1 For all $\mathscr{D}$,

$$\mathbb{P}(T^B_{\mathscr{D}}) = \mathbb{P}(T^Y_{\mathscr{D}}).$$

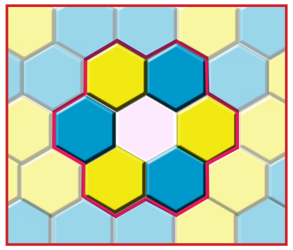
- We will in fact need the multiset version of Lemma 1.1 (i.e. $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$ and $T^Y_{\mathscr{D}_1 \ldots \mathscr{D}_k}$) but due to limitations of flower size, these cases do not present any additional difficulty.

- We condition on the petal configurations. Let η denote a petal configuration on $\mathscr{F}$ and $\overline{\eta}$ its color reverse. It is enough to show that (for all η)

$$\mathbb{P}(T^B_{\mathscr{D}} \mid \eta) = \mathbb{P}(T^Y_{\mathscr{D}} \mid \overline{\eta}).$$

- It may be assumed that all petals of $\mathscr{D}$ are already blue in η, η is not a trigger, and $\mathscr{D}$ is not already (blue) connected in η.

- Due to petal counting, $\mathscr{D}$ cannot have more than 3 components which are disconnected in η. The only possible case for $\mathscr{D}$ having 3 disconnected components is the alternating configuration, in which case the only possibility for connection is the pure state in the iris.



- We are down to the case of two separates components in η that need to be connected (we have implicitly absorbed all blue petals adjacent to $\mathscr{D}$ into $\mathscr{D}$). Here we have:
  - Micro-environment duality: in this (2–component) case, either all the blue petals of η are blue connected or the yellow petals of η are yellow connected.

- Therefore it suffices to consider the case of η having two non-adjacent blue petals which need to be connected and the rest of the petals all yellow.

- In the case of two non-adjacent blue petals and the rest of the petals yellow, there is exactly one mixed iris state which gives the required connection.
  - Thence $\mathbb{P}(T^B_{\mathscr{D}} \mid \eta) = a + s$ for this case, and with a similar result for yellow.

Given Lemma 1.1, the proof of Lemma 1 is almost immediate.

- First observe that if we let $\Pi_{\mathbf{r}\mathbf{r}'}$ denote the collection of path designates starting at $\mathbf{r}$ and ending at $\mathbf{r}'$, then

$$\kappa^B_{\mathbf{r}\mathbf{r}'} = \mathbb{P}(\bigcup_{\mathscr{P}\in\Pi_{\mathbf{r}\mathbf{r}'}} \mathscr{P}_B)$$

and similarly for $\kappa^Y_{\mathbf{r}\mathbf{r}'}$.

- Since the union is not disjoint, we will use inclusion-exclusion and prove equality on a term by term basis (note that $|\Pi_{\mathbf{r}\mathbf{r}'}| < \infty$).

- We need some notation: if $\mathscr{P}$ is a path designate, we write

$$\mathscr{P} = [H_{\mathbf{r}1}, (\mathscr{F}_1, h_1^e, h_1^x), H_{12}, (\mathscr{F}_2, h_2^e, h_2^x), H_{23}, \ldots, (\mathscr{F}_K, h_K^e, h_K^x), H_{K\mathbf{r}'}],$$

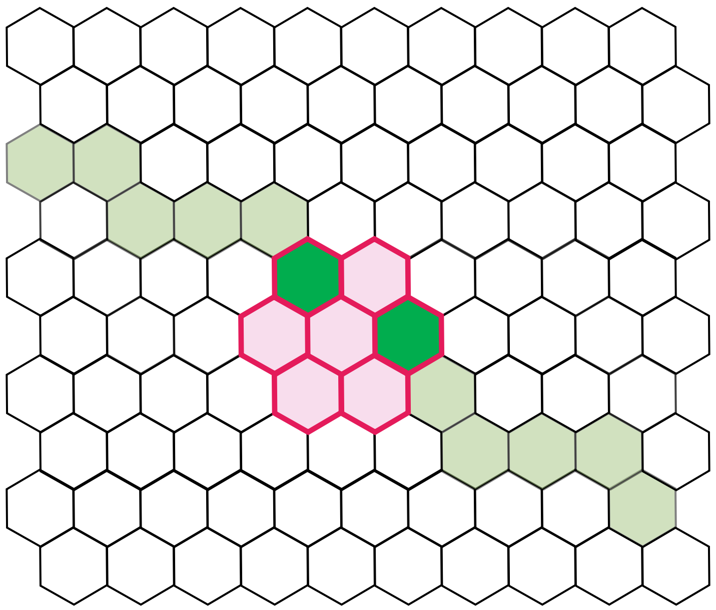
where $\mathscr{F}_1, \ldots, \mathscr{F}_K$ are flowers, h_j^e and h_j^x are entrance and exit petals, $H_{j,j+1}$ is a path in the complement of flowers connecting h_j^x to h_{j+1}^e, and $\mathbf{r}$ is used to denote the hexagon at $\mathbf{r}$, etc.

- Assume for simplicity each flower is used only once. Then we have

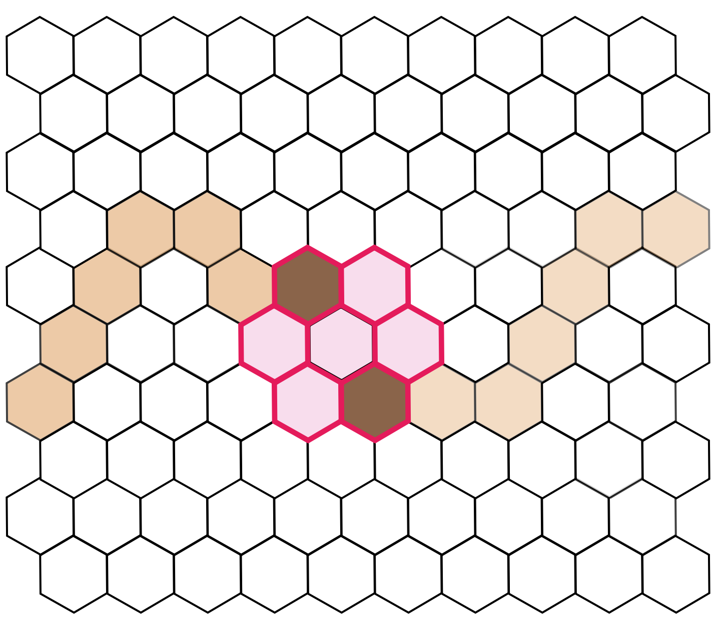
$$\mathbb{P}(\mathscr{P}_B) = \left(\frac{1}{2}\right)^{|H_{\mathbf{r},1}|} \mathbb{P}(T^B_{\{h_1^e,h_1^x\}}) \left(\frac{1}{2}\right)^{|H_{1,2}|} \ldots \ \mathbb{P}(T^B_{\{h_k^e,h_k^x\}}) \left(\frac{1}{2}\right)^{|H_{K,\mathbf{r}'}|}.$$

This is exactly equal to $\mathbb{P}(\mathscr{P}_Y)$ by Lemma 1.1.

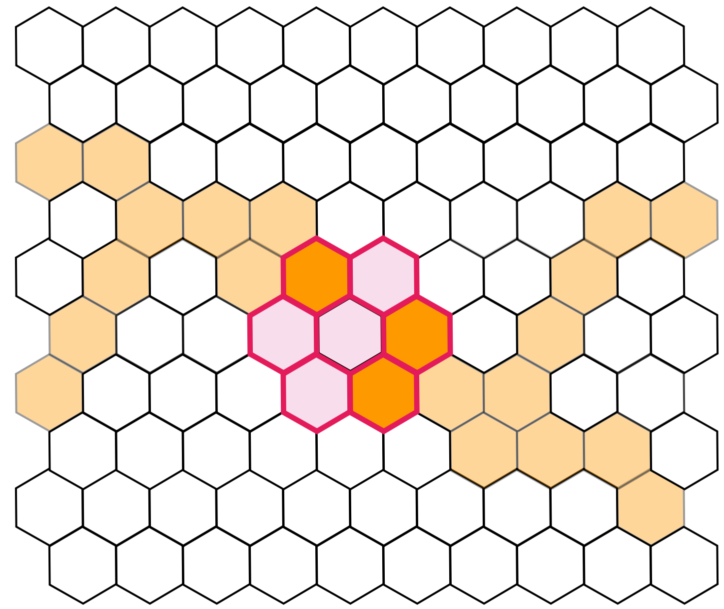
- For the more general case, note that multiple paths involving the same flower (or multiple visits to the same flower) have to be treated in one piece, i.e. we will need to consider multiple $\mathscr{D}$ sets. But this is exactly the content of Lemma 1.1; we are done.



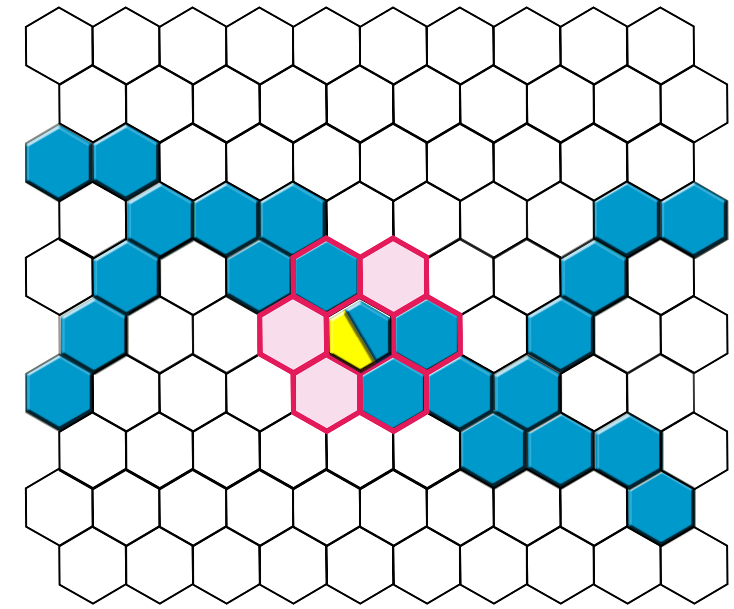
$\mathscr{P}_1, \mathscr{D}_1$



$\mathscr{P}_2, \mathscr{D}_2$



$\mathscr{P}_1 \cup \mathscr{P}_2 \quad \mathscr{D}_1, \mathscr{D}_2$



$(\mathscr{P}_1)_B \cap (\mathscr{P}_2)_B$

<u>Remark</u>: All this (along with assumption of periodic floral arrangement & $ae \geq 2s^2$) can be used to establish typical properties of 2*D* percolation systems which indicate critical behavior:

- No percolation of yellow or blue.
- Rings in annuli (with uniform probability) @ all scales.
- Power law bounds on connectivities.

But for us, this is just the beginning. We must face up to problem of color symmetry for transmissions in presence of *conditioned paths*.

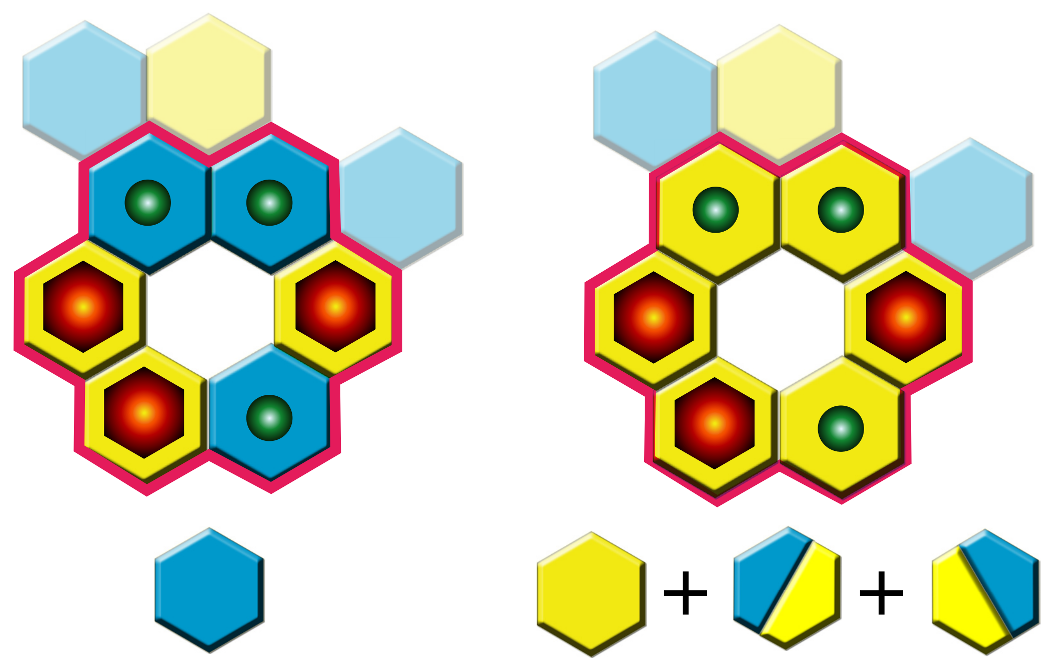
Recall that for CR we need to *condition* on two paths and change the color of a *third*. (All paths considered are supposed to be disjoint.) Previous strategy: exploit color symmetry on the level of flowers and (using designates) invoke an inclusion/exclusion type argument.

Problem: • Due to lack of color symmetry on small scales, *conditioning* will break the symmetry.

Example:

Conditioned Sites

Transmission Ports



$\frac{1}{2}$ (Trigger) $a + 2s$

Note: This is obvious in presence of triggering. But anyway would happen even without triggering.

• We can restore color symmetry by replacing the usual rules concerning disjointness with stochastically implemented $*$-*rules*:

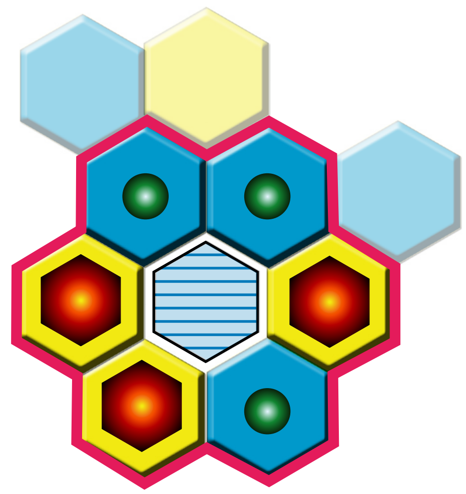
(1) When the color in question is at a disadvantage, allow certain (conditioned) petals to be shared.

- Triangular lattice (all monochrome hexagons can be split).
- Can view this as accommodating the collisions.

(2) When color in question is at an advantage, forbid from touching (i.e. using a hexagon adjacent to) petals already used by the conditioned set.

- Necessary (reason will be clear later) since we can only work with the color(s) of the conditioned set.

Previous Example



Always fine

with probability

$$\frac{s}{2(a+2s)}.$$

The $*$-*rules* will be implemented by random variables:

Definition Let $\mathscr{F}$ be a flower and $\diamondsuit$ a proper subset of the petals of $\mathscr{F}$. Let $\mathscr{D}$ be a set of petals on the complement of $\diamondsuit$. For each such $\diamondsuit, \mathscr{D}$ pair we have a 3-valued random variable $X_{\mathscr{D},\diamondsuit}$, which controls sharing of petals & close encounters and an additional 2-valued random variable $X^{\odot}_{\mathscr{D},\diamondsuit}$, which controls the sharing of irises.

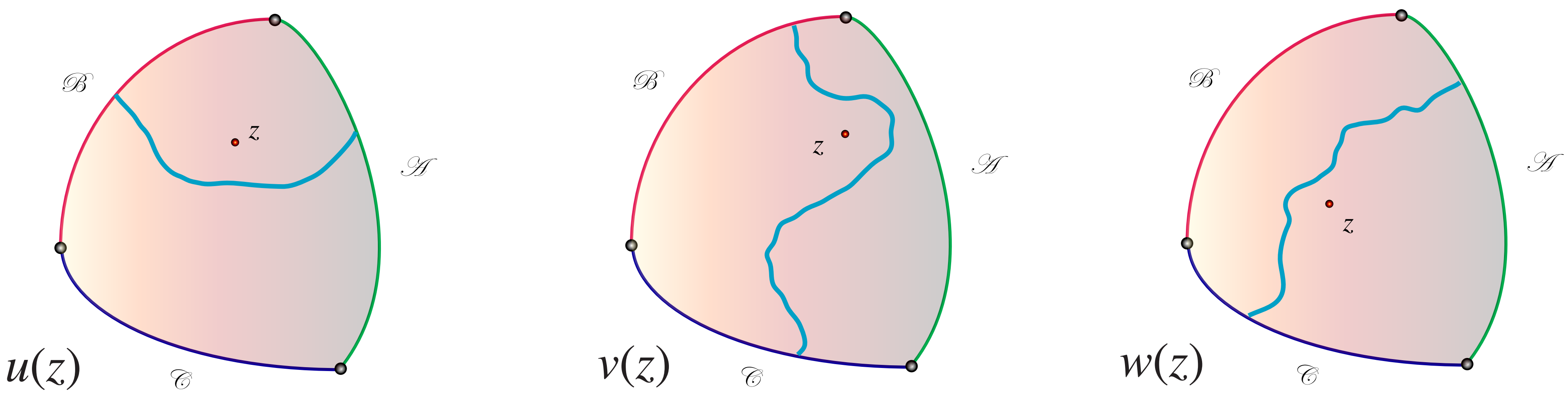
The ultimate (but not penultimate) lemma is that overall "balance" can be achieved (we shall omit the proof):

Lemma There exists a set of $*$-rules (laws for all relevant random variables) such that for any points x and y the probability of a $*$-transmission from x to y in the "complement" of any paths Γ_b & Γ_y is the same for blue as it is for yellow.

Here $*$-transmission means, depending on the values of the auxiliary random variables and the relevant colors involved, the possibility of leeway provided for the sharing of hexagons and/or adherence to no close encounter rules.

What does all this mean for our functions u_N, v_N and w_N? (N denotes lattice spacing of N^{-1})

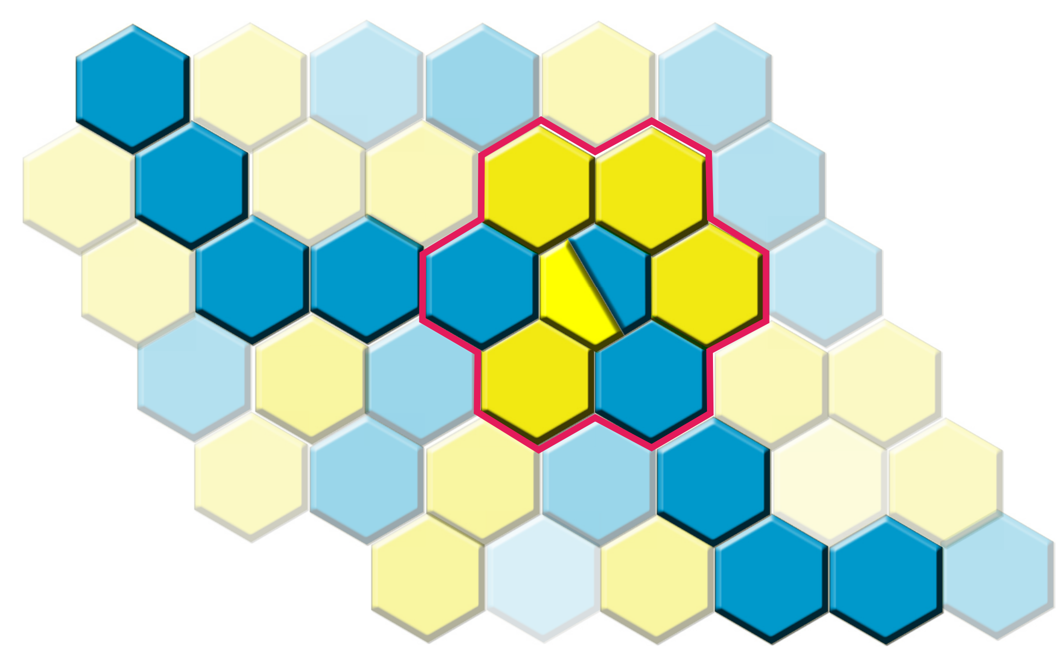
- Recall u_N, v_N and w_N represent probabilities of self-avoiding but possibly self-touching crossings.



- We will in fact prove CR–relations – and convergence to the appropriate harmonic functions – for modified lattice functions u_N^*, v_N^* and w_N^*.

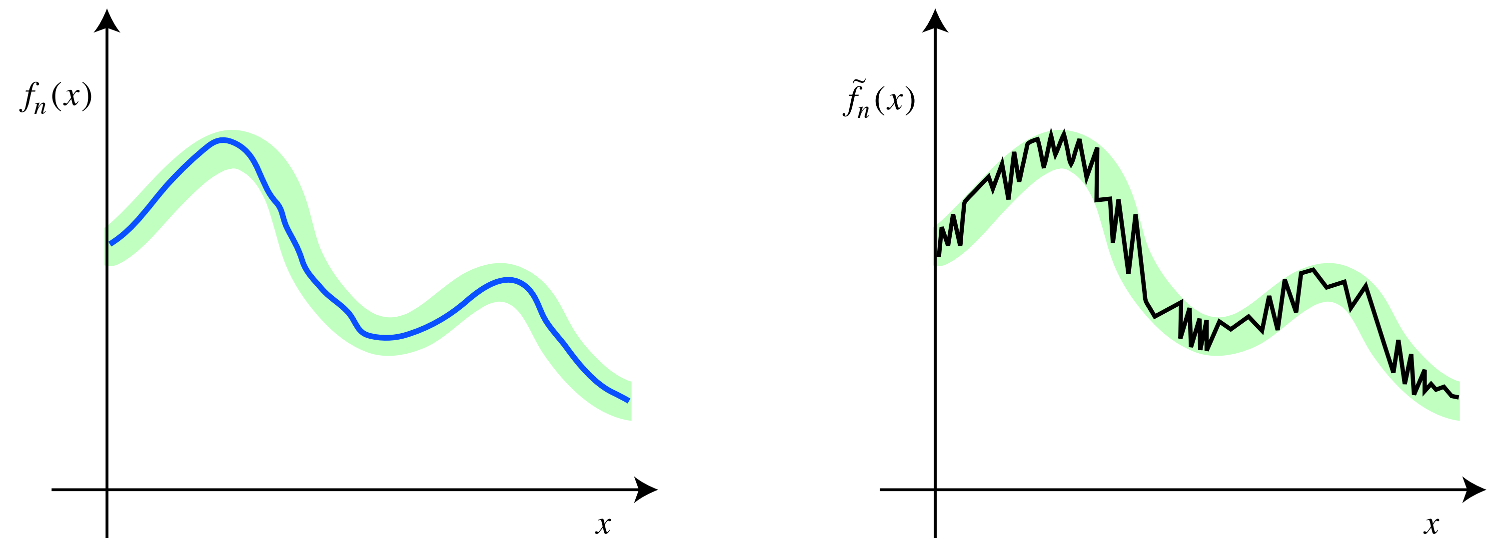
- In most cases, the $*$-version of the functions are simply indicators of the relevant events (where z is at the vertices of hexagons), *with all the $*$-rules taken into account*.

- However, when z is at the vertex of an iris hexagon and the path under consideration goes through the iris, then e.g. u_N^* is in fact the expectation of a random variable $\mathcal{U}_N^*(z)$, where $\mathcal{U}_N^*(z)$ is equal to 0 or 1 as before unless the iris is *pivotal* for the event in question to be achieved, in which case it is $\frac{1}{2}$ (e.g. the path would have led to a 1 had the iris been blue and a 0 had it been yellow).



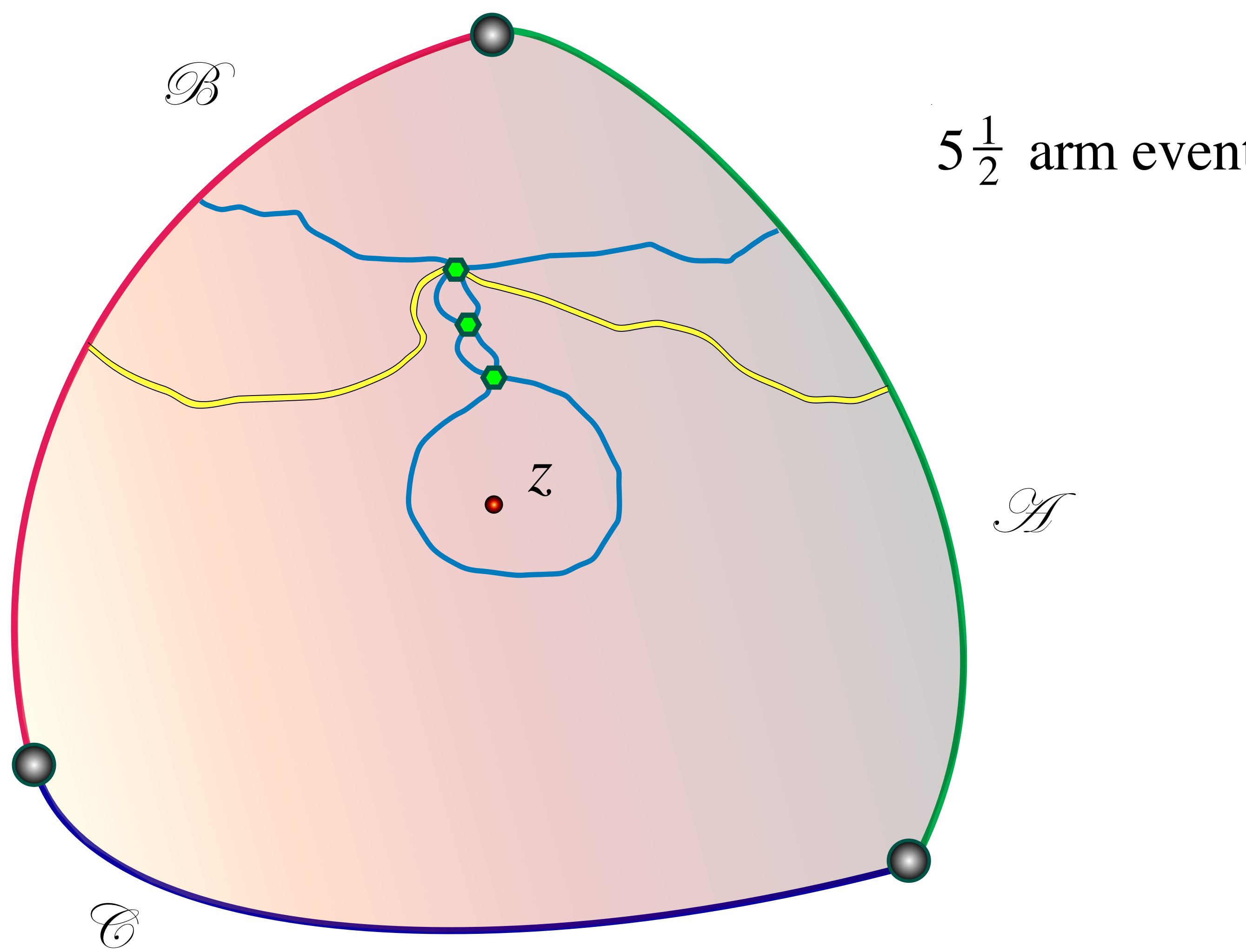
$\mathcal{U}_N^* \notin \{0,1\}$

Despite these seemingly drastic deviations from the original function (especially the derivative), we still have that, e.g. $|u_N - u_N^*| \to 0$ uniformly (on compact sets disjoint from the boundary) and hence recover the result for u, v, and w.



- Path which satisfy the "event" only come near z with vanishingly small probability.

- Any path of a configuration in $u_N^*(z)\Delta\, u_N(z)$ which is not close to z will lead to "five and a half" arms, the existence of which is also vanishingly small.



We now give a summary of certain technical difficulties we encounter (the price we pay for restoration of color symmetry).

I. FKG inequality and RSW lemmas.

- FKG was ostensibly difficult, but the assumption of $a^2 \geq 2s^2$ and the result in [CL] made it easy.
- For RSW, among other difficulties, had to actually read Kesten's book.
- Tragedy of RSW: lost rights to arbitrary floral arrangements.

II. Arms and Exponents.

- A five and a half arm argument, along with a three arm argument in the complement of a line segment was needed to show equivalence of Carleson-Cardy functions.
- Due to local correlations, standard KvB or Reimer's inequality does not apply, needed old fashioned conditioning argument.

III. Full Flower vs. "Used" Flower.

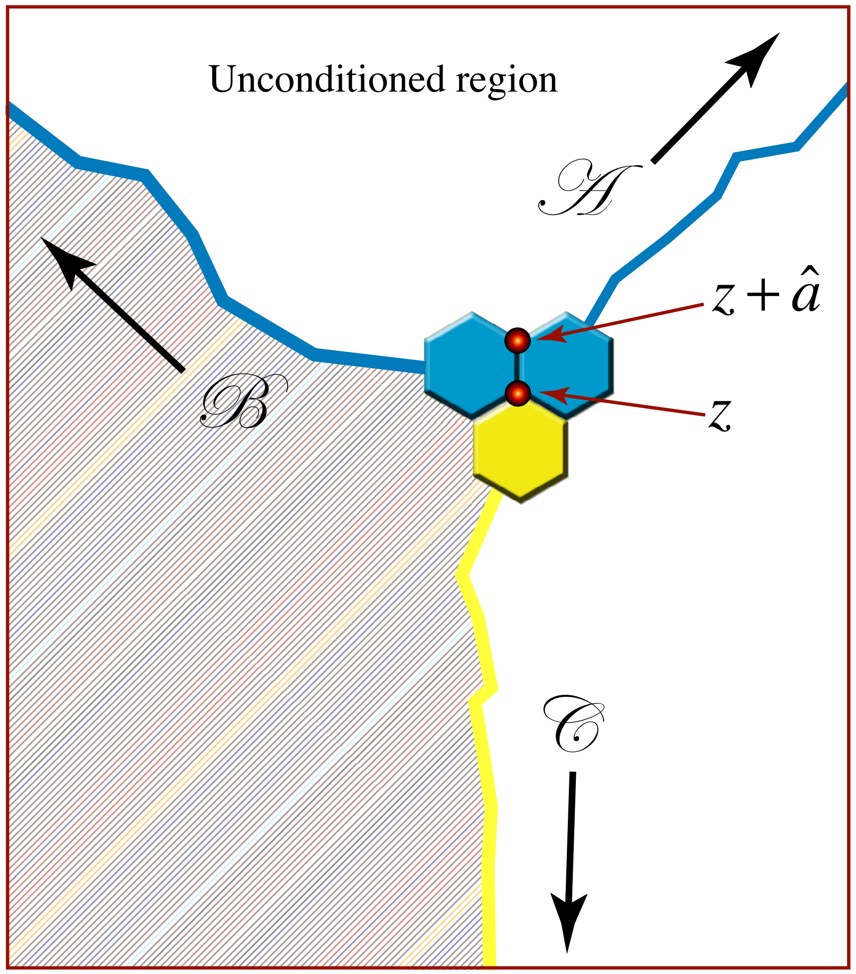
- This was needed in the conditioning argument in II.
- Seemingly "obvious", but involved meticulous and systematic consideration of all possibilities.

IV. The Iris in Cauchy-Riemann Switch.

- No sensible mechanism to have path designate start @ iris. CR–relations require effort.

V. Producing the Lowest Path for Conditioning (loop erasure).

To actually switch color, recall that we condition on some “lowest” path.



- We must be able to unambiguously partition the space of configurations with the events “Γ_i is the lowest path”. To this end, we need to ensure some paths are self-avoiding and non-self-touching (*strictly self-avoiding*).
- However, recall that not all our paths are strictly self-avoiding. Geometrically it is clear that any path can be turned into a strictly self-avoiding path by deleting all loops. In our context, there are two problems:

  (1) We must still keep all loops which “capture” z.

  (2) The random variables may cause certain paths to be “dumped” after deletion of loops.

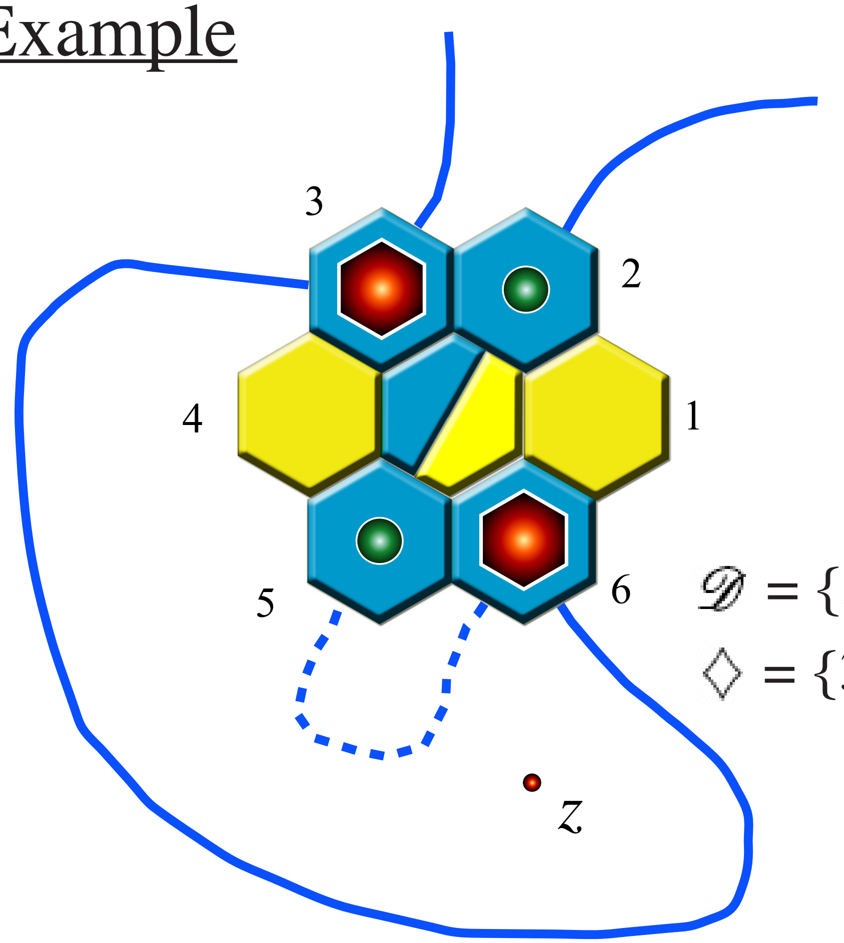
More precisely, to decide whether some ω is in $\mathcal{U}_N^*$, we check *all* paths in ω which satisfy the geometric criterion of connecting $\mathscr{A}$ to $\mathscr{B}$ and separating z from $\mathscr{C}$ against the random variables $X_{\mathscr{D},\diamondsuit}$, so suppose the path Γ visits a flower $\mathscr{F}$ multiple times:

- The first time through the flower is “free”
- For the second pass through the flower, the portion of flower used by first pass now defines the conditioned set $\diamondsuit$, which sets the values of $X_{\mathscr{D},\diamondsuit}$. Similarly for further passes.
- Hence unless the path receives the proper “permissions” from all the relevant $X_{\mathscr{D},\diamondsuit}$, it is useless for achieving the event.

It is not a priori clear that if Γ in ω achieves the event, then the reduced version $\hat{\Gamma}$ in ω also achieves the event, since loop erasure may mean changing the conditioned sets $\Diamond$ in one or more passes through flowers.

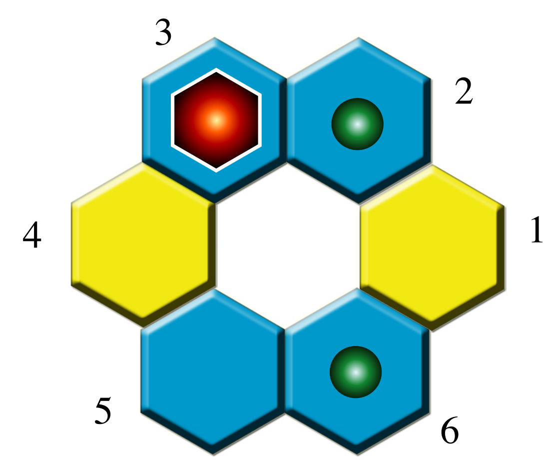
In fact, the statement that the fully reduced version of Γ (i.e. all loops erased except for the one necessary for "capture" of z) satisfies the event is false:

Example



$\mathscr{D} = \{5,2\}$

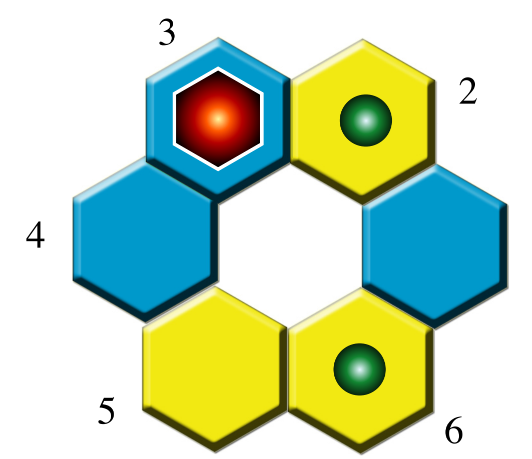
$\Diamond = \{3,6\}$



$\mathscr{D} = \{2,6\}$

$\Diamond = \{3\}$

Transmission: $(a + 2s)$



Trigger

Transmission: $\frac{1}{2}$

A partial result of the type needed in fact does turn out to be true, and this is all we need.

Definition. We define the last *lasso point* of a path to be a shared hexagon or a close encounter pair which is part of a relatively simple closed loop of the path with z in its interior. We have similar definitions for the next to last lasso point, etc.

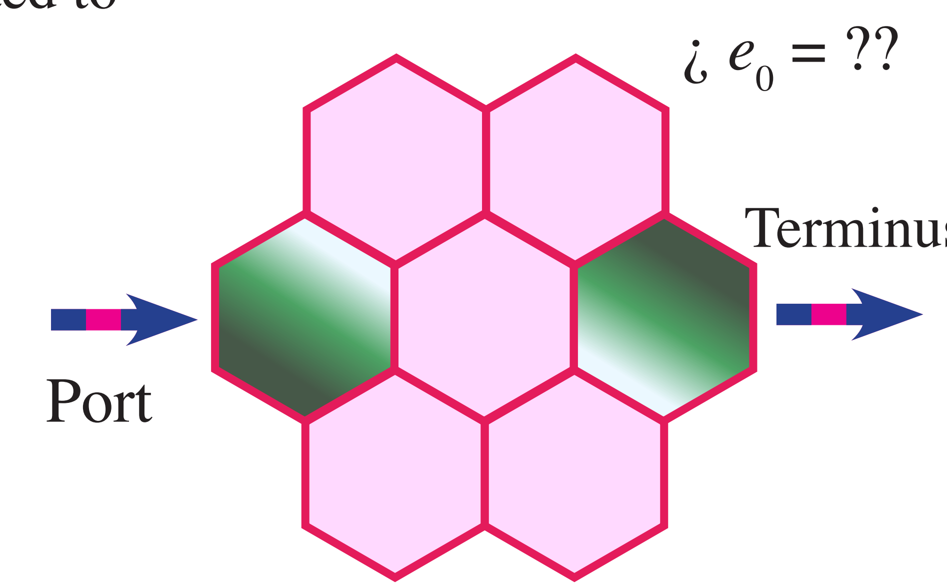
Lemma. Suppose ω is in $\mathcal{U}_N^*$. Then in ω there is a path satisfying the requirements of $\mathcal{U}_N^*$ such that the part of the path from $\mathscr{A}$ to the last lasso point necessary for the capture of z can be regarded as having no sharings and no close encounters with itself.

We shall not prove all of this, but only demonstrate one simple case that one must consider. Again it is enough to consider what happens in a single flower.

Let $\mathscr{F}$ be a flower. Let e_0 denote the 1st petal visited by path in its first pass through $\mathscr{F}$. Let c denote the last petal of $\mathscr{F}$ visited by path before capture of z. Must show $e_0 \rightarrow c$ reduces to a strictly self–avoiding path.

There is nothing to prove unless a transmission through the iris is required to get to c. We call the first petal visited by the path in this transmission port and the last petal the terminus. Case considered here: Single transmission with no hexagon shared and with the port & terminus diametrically opposed.

- e_0 cannot be *equal* to the terminus, because then transmission is not actually needed in the reduced path (since capture of z is purported to take place after the transmission).
- Similarly e_0 cannot be a neighbor of the terminus.
- But if e_0 is the port then in the reduced path we simply have a direct (unconditioned) transmission to the terminus and no random variable will be involved.
- Similarly for a neighbor of the port.



We are out of petals. ▨

We are done in this (almost trivial) case. There are many more cases that need to be considered, in particular *the case where a petal is shared by the unreduced path*.

(I) Wrap–Up. After much work, result is that lattice functions u_N, v_N & w_N for *this* model converge to the "Cardy–Carleson" functions.

I.e. the same result as for triangle site lattice model.

Pretty much a complete proof that the continuum limits of both systems are exactly the same; Reasonable and fairly robust statement of *universality*.

Central dogma for theory of critical phenomona since the 1960's.

(II) Limitations

(a) Not a standard (well known) percolation model.

(b) Within context of model, didn't get most complete result.

Although model does indeed have parameters –"some generality".

(c) Aside from some practical (and technical) considerations, did *not* learn much about the nature of and convergence to continuum limit – beyond what was already known.