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Percolation and ferromagnetism on $\mathbb{Z}^2$: the q-state Potts cases[1]

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Abstract

For the nearest-neighbor Potts ferromagnets on $\mathbb{Z}^2$, it is shown that the magnetized states are characterized by percolation of the preferred species, which generalizes a result known previously only for the Ising system.

Keywords: Percolation; Potts models; Random cluster models; FKG inequalities

The connection between phase transitions and underlying geometric phenomena – usually percolation – has been the subject of innumerable studies. In this note, attention is focused on two-dimensional problems where this connection is unusually sharp. It is assumed that the reader is familiar with the basic tools of the trade: Elementary spin-systems and their Gibbs states, random cluster representations, FKG properties and the various notions of connectivity for $\mathbb{Z}^2$. For relevant background on these topics, see Georgii (1989) for spin-systems, FKG properties and phase transitions, see Grimmett (1989) for information about percolation processes in general and see Grimmett (1994, 1995) for material on the Potts model and the random cluster model in particular.

A number of years back, Coniglio et al. (1976, 1977) showed that the low-temperature phase of the nearest-neighbor Ising ferromagnet on $\mathbb{Z}^2$ was *characterized* by percolation; the infinite volume $\oplus$-state is distinguishable from the corresponding $\ominus$-state iff there is an infinite cluster of $\oplus$-spins in the former. Explicitly, if the temperature is so high that the $\oplus$-spins fail to percolate in the $\oplus$-state then there is only one limiting Gibbs state and whenever there is an infinite $\oplus$-cluster in the $\oplus$-state, the spontaneous magnetization is positive. It is noted, however, that this sort of characterization is almost certain *not* to hold in higher dimensions or even on other (non-planar) two-dimensional graphs. It is, for better or worse, a peculiar feature of the system at hand.

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The core of the derivation, and its success vs. failure on other graphs is uniqueness. In the above cited, it was shown that if, in the plus state, there is an infinite connected cluster of $\oplus$-spins, this cluster is unique (with probability one) and (with probability one) precludes even the possibility of $*$-percolation of minuses. This is unlikely to be true under general conditions and, to be definitive, fails at infinite temperatures on $\mathbb{Z}^d$ if $d > 2$ (Campanino and Russo, 1985) and at finite temperature in sufficiently high dimension (Aizenman et al. 1987).

What is the status of the Potts ferromagnets? Under fairly general conditions, it was shown in Aizenman et al. (1987, 1988) that percolation in the random cluster (FK) representation is the necessary and sufficient condition for low-temperature behavior. Thus, e.g. for the nearest-neighbor models on $\mathbb{Z}^d$, there is certainly an infinite cluster of the preferred species. However, to amplify the above mentioned, in high dimension, percolation of spins is likely to occur in the high-temperature phase. Worse yet, there could even be infinite clusters of the subdominant species in the low-temperature phase near the transition temperature as is the case in the Ising systems (Aizenman et al., 1987). Thus, with the exception of two dimensions, percolation in the FK representation may well represent the limits of the percolation/phase transition picture for these systems, unless (as is the case for $q \gg 1$) the transition is strongly first order.

However, there is every reason to believe that the CNPR picture holds for two-dimensional problems in some generality. The purpose of this note is to show that it goes at least as far as Potts models. It is remarked that this set of issues is, in fact, of some practical importance in numerical simulations. Indeed, it is believed that this effect is responsible for the anomalous scalings that are observed in simulations of two-dimensional systems using the IC algorithm (cf. Machta et al. for a discussion of this point).

For the Potts models, some preliminary steps have already been taken. In Giacomin et al. (1995) it is shown that below the transition temperature in the magnetized states, none of the subdominant species, nor any combination thereof, manage to percolate. Further, for $q \gg 1$, the derivation of Kotecký and Shlosman (1982) – and any number of subsequent papers, e.g. Laanait et al. (1991) – can be used to show that at or below the transition temperature, the infinite cluster is so dense that it all but precludes the subdominant species, let alone allow them to percolate.

Of both general and specific relevance is the seminal result of Gandolfi, Keane and Russo – the foundation upon which everything in this note rests. In Gandolfi et al. (1988), a large class of dependent site percolation problems on $\mathbb{Z}^2$ are considered. It is assumed that the measure is

(1) ergodic and invariant under translations,
(2) invariant under axis reflections and
(3) satisfies the FKG condition.

Under (1)–(3), if there is percolation, then with probability one, the infinite cluster is unique and all vacant sites lie in finite $*$-connected clusters.

On the basis of the construction of the "wired" states in Aizenman et al. (1988), it is easy to show that the random cluster measures *or* the associated Gibbs measures satisfy the requisite invariances (1) and (2). But obviously, something is missing:

(a) If $q > 2$, we are not dealing with a binary problem.

(b) Even if the binary aspect were not a requirement, it is clear that among the q separate elements of the spin-space, there is no natural way to order the states.

However, the following scheme, that simultaneously circumvents (a) and (b), is naturally suggested: single out a particular species – hereafter referred to as "green" and regard all other colors as equivalent (grey). Specifically, if $\mathscr{G}$ is a finite graph and $\sigma_{\mathscr{G}}$ is a spin configuration (assignment of one of the q colors to each site of $\mathscr{G}$) define

$$\sigma_{\mathscr{G}} \sim \sigma'_{\mathscr{G}} \text{ if } \sigma_{\mathscr{G}} \text{ and } \sigma'_{\mathscr{G}} \text{ have the same green sites.} \tag{1}$$

If $\mu_{\mathscr{G}}(-)$ is a probability measure on $\{1, 2, \ldots, q\}^{\mathscr{G}_v}$ – where $\mathscr{G}_v$ denotes the sites of $\mathscr{G}$ – we may define, in a natural way, a probability measure on equivalence classes, $\eta_{\mathscr{G}}$ of configurations:

$$\nu_{\mathscr{G}}(\eta_{\mathscr{G}}) = \sum_{\sigma_{\mathscr{G}} \in \eta_{\mathscr{G}}} \mu_{\mathscr{G}}(\sigma_{\mathscr{G}}). \tag{2}$$

We have just solved problems (a) and (b) and it would seem plausible that for ferromagnetic Potts models, such a "reduced" measure will satisfy condition (3). This is demonstrated below.

Lemma. *Let $\mu_{\mathscr{G}}$ denote the Gibbs measure on a finite graph $\mathscr{G}$ corresponding to the ferromagnetic q-state Potts Hamiltonian*

$$\mathscr{H} = -\sum_{\langle i,j\rangle \in \mathscr{G}e} J_{i,j}\delta_{\sigma_i\sigma_j},$$

where $\mathscr{G}_e$ denotes the bonds of $\mathscr{G}$ and with $\infty > J_{i,j} \geqslant 0\ \forall \langle i,j\rangle \in \mathscr{G}_e$. (For simplicity, the case of an external field is not discussed.) Then the measure $\nu_{\mathscr{G}}(-)$ reduced, as defined in Eq. (2), from the canonical Gibbs measure of the Hamiltonian $\mathscr{H}$ at temperature $1/\beta$ is (strong) FKG.

Proof. In the following, let us consider a fixed $\mathscr{G}$ once and for all and omit all the $\mathscr{G}$ subscripts. Further, taking advantage of the binary nature of the problem, when no confusion will arise, let us allow "configurations" η to serve the dual notational function of subsets of $\mathscr{G}_v$ as well as equivalence classes of spin configurations.

For a proof of this lemma, it is required to show that the FKG lattice condition is fulfilled; that is if η_1 and η_2 are configurations, we need that $\nu(\eta_1 \wedge \eta_2)\nu(\eta_1 \vee \eta_2) \geqslant \nu(\eta_1)\nu(\eta_2)$. As is well known, this can be accomplished by the comparison of configurations that differ only at a single site, i.e. starting from $\zeta^{[0]} = \eta_1 \wedge \eta_2$, consider a sequence $\zeta^{[0]}, \zeta^{[1]}, \ldots, \zeta^{[n]} = \eta_1, \zeta^{[0]} \prec \zeta^{[1]} \cdots \prec \zeta^{[n]}$ where successive members of the sequence differ by the addition of a single site from η_1 that is not in η_2. Thus, under the assumption that for some η,

$$\nu(\eta \wedge \eta_2)\nu(\eta \vee \eta_2) \geqslant \nu(\eta)\nu(\eta_2), \tag{3}$$

it is sufficient to verify that Eq. (3) holds with η replaced by an $\eta' = \eta \vee k$, where k is an arbitrary "site" that is not in η_2.

Let us examine the structure of these probabilities $\nu(\eta)$: Aside from an overall normalization constant, there is a Boltzmann factor weighting the (energetically favorable) pairings of the sites in η, no interaction between the sites in η and $\mathscr{G}\backslash\eta$ and a term that accounts for the weight of all the equivalent spin configurations on $\mathscr{G}\backslash\eta$. The former is explicitly computed, namely $e^{\beta\Phi(\eta)}$, where

$$\Phi(\eta) = \sum_{i,j\in\eta} J_{i,j}. \tag{4}$$

The latter term is exactly the $(q-1)$-state Potts partition function for the sites on $\mathscr{G}\backslash\eta$ with ferromagnetic pair interactions inherited from the original Hamiltonian. These considerations are summarized by the formula

$$\nu(\eta) \propto e^{\beta\Phi(\eta)} Z^{q-1}_{\mathscr{G}\backslash\eta}. \tag{5}$$

Let us attend to the desired inequality. Making use of the assumed Eq. (3), it is easy to see that

$$\begin{aligned}
&\nu(\eta' \wedge \eta_2)\nu(\eta' \vee \eta_2) - \nu(\eta')\nu(\eta_2) \\
&= \nu(\eta \wedge \eta_2)\nu(\eta \vee \eta_2)\left[\frac{\nu(\eta' \vee \eta_2)}{\nu(\eta \vee \eta_2)}\right] - \nu(\eta)\nu(\eta_2)\left[\frac{\nu(\eta')}{\nu(\eta)}\right] \\
&\geqslant K\left[\frac{\nu(\eta' \vee \eta_2)}{\nu(\eta \vee \eta_2)} - \frac{\nu(\eta')}{\nu(\eta)}\right]
\end{aligned} \tag{6}$$

with K positive. The result follows if it can be shown that the final quantity in the square brackets cannot be negative.

Now the energy factors for η and η' differ only by the terms involving the site k:

$$\Phi(\eta') - \Phi(\eta) = \sum_{i\in\eta} J_{i,k}. \tag{7}$$

Thus, $e^{[\Phi(\eta')-\Phi(\eta)]} \leqslant e^{[\Phi(\eta' \vee \eta_2)-\Phi(\eta \vee \eta_2)]}$ since the latter is summed over a bigger set. It is therefore sufficient to show that $Z^{q-1}_{\mathscr{G}\backslash\eta}/Z^{q-1}_{\mathscr{G}\backslash\eta'}$ is larger than $Z^{q-1}_{\mathscr{G}\backslash\eta \vee \eta_2}/Z^{q-1}_{\mathscr{G}\backslash\eta' \vee \eta_2}$. For simplicity, let us now consider matters from the perspective of the portion of $\mathscr{G}$ that remains. Thus let $\mathscr{D} \subset \mathscr{G}$ denote, e.g. $\mathscr{G}\backslash\eta'$ (so that $k \notin \mathscr{D}$) and $\mathscr{D}^{\#}$ denote $\mathscr{D}$ together with $\{k\}$ and all the bonds that connect k to $\mathscr{D}$. Notice that

$$\frac{Z^{q-1}_{\mathscr{D}^{\#}}}{Z^{q-1}_{\mathscr{D}}} = \sum_{\sigma_k}\left\langle \exp\left[\beta \sum_{j\in\mathscr{D}} J_{jk}\delta_{\sigma_k\sigma_j}\right]\right\rangle_{\mathscr{D},q-1}, \tag{8}$$

where it is understood that the σ_j's are now $q-1$ state variables and $\langle - \rangle_{\mathscr{D},q-1}$ denotes expectation with respect to the $q-1$ state Potts measure as defined on $\mathscr{D}$ via the Hamiltonian (described in the statement of this lemma) restricted to $\mathscr{D}$. In the expression on the right-hand side of Eq. (8), it is seen that the object σ_k has now taken on the status of an (uncoupled) auxiliary variable. Let us expand the exponential á la Fortuin and Kasteleyn:

$$e^{\beta J_{jk}\delta\sigma_j\sigma_k} = R_{j,k}\delta_{\sigma_j\sigma_k} + 1 \tag{9}$$

with $R_{j,k} = e^{\beta J_{jk}} - 1$. The individual factors in the product over the bonds may be expanded out: $1 + \sum_{j\in\mathscr{D}} R_{j,k}\delta_{\sigma_j\sigma_k} + \sum_{j,j'\in\mathscr{D},j'\neq j} R_{j,k}R_{j',k}\delta_{\sigma_j\sigma_k}\delta_{\sigma_{j'}\sigma_k} + \cdots$ and, with the exception of the first term, an individual term is only non-zero when all of the participating σ_j's agree with σ_k – and hence each other. However, in these expressions, σ_k is decoupled so the particular value of σ_k is of no importance. When the actual summation over σ_k is performed, the expectation of the product of the δ-functions that appear in each (non-trivial) term will therefore be the probability that all of the relevant σ_j's agree. To formalize these notions, let $\mathscr{A}_k \equiv \mathscr{A}_k(\mathscr{D})$ denote all the (site) subsets of $\mathscr{D}$ that are needed for the evaluation of Eq. (8):

$$\mathscr{A}_k = \{A \subset \mathscr{D} \mid R_{j,k} \neq 0 \; \forall j \in A\}. \tag{10}$$

It is seen that

$$\frac{Z_{\mathscr{D}\#}}{Z_{\mathscr{D}}} = q - 1 + \sum_{A \in \mathscr{A}_k} R_A \langle \delta_{\sigma_A} \rangle_{\mathscr{D}, q-1}, \tag{11}$$

where $R_A = \prod_{j \in A} R_{j,k}$ and δ_{σ_A} is one if σ_j is constant throughout A and zero otherwise. Notice that the notation is consistent even if A is a singleton; in these cases, the relevant expectation is one. (For $A = \emptyset$, it is declared that $R_A = 0$.)

To evaluate the less trivial cases, it is useful to go over to the FK representation. As was pointed out in Aizenman et al. (1989, 1988), the expectation of any observable in the spin language may be expressed, in the random cluster language, as weighted sums of probabilities of the various cluster connectivites on the support of the observable. The case at hand is particularly easy. Let $\mathbb{P}_{\mathscr{D},q-1}(-)$ and $\mathbb{E}_{\mathscr{D},q-1}(-)$ denote the random cluster probability and expectation corresponding to the $(q-1)$ state Potts model on the graph $\mathscr{D}$ with the described Hamiltonian. For $A \subset \mathscr{D}$, let $|C(A)|$ denote the number of separate components of A in a random cluster configuration and let $|A|$ denote the number of sites in A. Then

$$\begin{aligned}\langle \delta_{\sigma_A} \rangle_{\mathscr{D}} &= \sum_{n=1}^{|A|} \frac{1}{(q-1)^{(n-1)}} \mathbb{P}_{\mathscr{D},q-1}(|C(A)| = n) \\ &= (q-1)\mathbb{E}_{\mathscr{D},q-1}\left(\left[\frac{1}{q-1}\right]^{|C(A)|}\right).\end{aligned} \tag{12}$$

The key observation, which has been the goal of all these manipulations, is that the function $[1/(q-1)]^{|C(A)|}$ is increasing. (Because $|C(A)|$ is manifestly decreasing and because $1/(q-1) < 1$.) With this in mind, let us compare $Z_{\mathscr{D}\#}^{q-1}/Z_{\mathscr{D}}^{q-1}$ with $\mathscr{D} = \mathscr{G}\backslash\eta$ and with $\mathscr{D} = \mathscr{G}\backslash\eta \vee \eta_2$. In the case where $\mathscr{D}$ is the bigger set, namely $\mathscr{D} = \mathscr{G}\backslash\eta$, there are, first and foremost, more (positive) terms of the form in Eq. (11) to be evaluated. (Explicitly: $\mathscr{A}_k(\mathscr{G}\backslash\eta) \supset \mathscr{A}_k(\mathscr{G}\backslash\eta \vee \eta_2)$.) Let us neglect this advantage and focus on the terms that are present in both expansions. Obviously, the R_A's are the same and, due to the FKG property of the random cluster measure, the desired result is seen to follow. Indeed, if $\tilde{\mathscr{B}} \supset \mathscr{B}$ and all the couplings on the common graph are the same it follows, for $s \geqslant 1$, that $\mathbb{P}_{\tilde{\mathscr{B}},s} \underset{\text{FKG}}{\geqslant} \mathbb{P}_{\mathscr{B},s}$. (To see this, observe that if $s \geqslant 1$, both measures are FKG and that $\mathbb{P}_{\mathscr{B},s}$ may be regarded as $\mathbb{P}_{\tilde{\mathscr{B}},s}$ conditioned on the event that certain bonds are vacant – a decreasing event.) Thus, term by term, a larger contribution is incurred for $\mathscr{D} = \mathscr{G}\backslash\eta$ and the conclusion is that

$$\frac{Z^{q-1}_{\mathscr{G}\backslash\eta' \vee \eta_2}}{Z^{q-1}_{\mathscr{G}\backslash\eta \vee \eta_2}} \geqslant \frac{Z^{q-1}_{\mathscr{G}\backslash\eta'}}{Z^{q-1}_{\mathscr{G}\backslash\eta}} \tag{13}$$

which completes the argument. □

Remark. By definition, the strong FKG property means (or implies) that the measures conditioned on any cylinder event are themselves FKG. Thus, e.g. on finite subsets of $\mathbb{Z}^2$, if all the boundary sites are set to green, the resultant measure, the finite volume green state, is FKG. Furthermore, this measure FKG dominates a similar measure obtained by any other choice of boundary conditions.

The percolation property of the Potts models on the square lattice is obtained as an immediate corollary.

Theorem. *For the ferromagnetic Potts models on $\mathbb{Z}^2$ with isotropic interactions, the CNPR result holds: If there is no infinite cluster of "greens" in the limiting "green" state, the Gibbs state is unique while if the green state has an infinite cluster, the system is below or, perhaps at, the transition temperature (here defined as the infimun of the temperatures where the spontaneous magnetization vanishes). Finally, if at the transition temperature there is order/disorder coexistence (which here means the existence of at least $q + 1$ extermal Gibbs states q of which have non-vanishing magnetization and one of which has zero magnetization) – which occurs if q is sufficiently large – then there is an infinite cluster of greens in the green state and similarly for the other colors while in the limiting state arising from free boundary conditions, there is no percolation of any single color.*

Proof. As we have just learned, in finite volume, the maximal (as in green-most) measures that are temperature $1/\beta$ Gibbs states of the appropriate Hamiltonian are obtained by setting all the spins on the boundary to green. As it happens, this coincides with the wired measure of the FK representation so for either reason, all the requisite invariances needed for Gandolfi et al. (1988) are satisfied in these limiting states. If there is no percolation of green spins in the green state, there cannot be percolation in the FK representation in the wired state and we must be in the phase with a single Gibbs state. On the other hand, if there is percolation of greens in the green state then, using the core result of Gandolfi et al. (1988), the grey spins – and hence all other colors – are, with probability one, confined to finite clusters. Thus the green state is distinguished from, e.g. the blue state. Thence we *are* in the multiple phase regime which in turn implies that the FK percolation density is positive and this, of course, is exactly the spontaneous magnetization.

Finally, when there is no percolation in the free boundary random cluster measures it follows that the underlying Gibbs measure is ergodic and has all the underlying invariances. (The absence of percolation in the free state along with the presence of an infinite cluster in the wired state is the signature of phase coexistence.) For the spin-system, the presence of green infinite clusters implies the presence of other colored infinite clusters which, back in the binary problem, would imply the simultaneous existence of a green and grey infinite cluster. Evidently, when there is no FK percolation in the free state, there is no colored percolation in the Gibbsian version of the free state. □

Remark. One can consider more general ferromagnetic Potts Hamiltonians of the form

$$\mathscr{H} = -\sum_{B \subset \mathscr{G}} J_B \delta_{\sigma_B}, \tag{14}$$

where δ_{σ_B} was defined following Eq. (11) and with $J_B \geqslant 0$ for every B. (Here the graph structure of $\mathscr{G}$ is irrelevant; for the purpose of this remark, $\mathscr{G}$ is just a collection of sites.) It is not difficult to see that the analog of the FKG lemma holds in these cases. The principal steps are identical; there is a notational obstruction in and of the fact that various sets A could appear multiple times – for different reasons – in the analog of Eq. (11). Thus, a name for the coefficients that is more clever than R_A would have to be concocted.

Now if $\mathscr{H}$ is defined on $\mathbb{Z}^2$ with B's that are translation and reflection invariant, the Gandolfi et al. (1988) result holds e.g. for the green states. However, the CNPR picture could fail because percolation of green spins may not be required for spontaneous magnetization. Indeed, in the FK representation, the system reduces to a percolation problem involving the amalgamation of the various objects representing the sets B, where $J_B \neq 0$, along with the notion of connectedess defined by non-empty intersection of the site content of such objects. Such infinite clusters do not imply a $\mathbb{Z}^2$-connected infinite cluster of underlying sites and it is easy to cook up examples where there is positive magnetization but no percolation. However, under the additional assumption that all of the B's in Eq. (14) are connected sets, the extension of the above theorem to the more general Potts case is immediate.

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