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Convergence to Equilibrium of Random Ising Models in the Griffiths Phase

K. S. Alexander,[1] F. Cesi,[2] L. Chayes,[3] C. Maes,[4] and F. Martinelli[5]

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We consider Glauber-type dynamics for disordered Ising spin systems with nearest neighbor pair interactions in the Griffiths phase. We prove that in a nontrivial portion of the Griffiths phase the system has exponentially decaying correlations of distant functions with probability exponentially close to 1. This condition has, in turn, been shown elsewhere to imply that the convergence to equilibrium is faster than any stretched exponential, and that the *average over the disorder* of the time-autocorrelation function goes to equilibrium faster than $\exp[-k(\log t)^{d/(d-1)}]$. We then show that for the diluted Ising model these upper bounds are optimal.

KEY WORDS: Random spin systems; diluted Ising model; Glauber dynamics; relaxation time; Griffiths singularities; FK representation.

We consider a random Ising model with formal Hamiltonian

$$H(\sigma) = -\sum_{\langle x, y\rangle} J_{xy}\sigma(x)\,\sigma(y) \tag{1.1}$$

The underlying lattice is $\mathbb{Z}^d$ and the spin variables $\sigma(x)$ take values ± 1. The notation $\langle x, y\rangle$ means that the sum is taken over all pairs of nearest

[1] Department of Mathematics, DRB 155, University of Southern California, Los Angeles, California 90089-1113; e-mail: alexandr@math.usc.edu

[2] Dipartimento di Fisica, Università "La Sapienza," 00185 Rome, Italy; and INFM Unità di Roma, Rome, Italy; e-mail: cesi@zephyrus.roma1.infn.it

[3] Department of Mathematics, University of California, Los Angeles, California 90095-1555; e-mail: lchayes@math.ucls.edu

[4] Instituut voor Theoretische Fysika, K.U. Leuven, B-3001 Leuven, Belgium; and Onderzoeksleider F.W.O. Flanders, Belgium; e-mail: christian.maes@fys.kuleuven.ac.be

[5] Dipartimento di Energetica, Università dell' Aquila, Italy; e-mail: martin@mat.uniroma.3.it

neighbors (pairs of sites whose *euclidean distance* is one). We denote by $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ the *configuration space*. For $x \in \mathbb{Z}^d$ we let $|x| = \max_{i \in \{1,..., d\}} |x_i|$. The associated distance function is denoted by $d(\cdot, \cdot)$. By Q_L we denote the cube of all $x = (x_1,..., x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0,..., L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball of radius L centered at the origin, i.e., $B_L = Q_{2L+1}((-L,..., -L))$. A subset Λ of $\mathbb{Z}^d$ is said to be a *multiple* of Q_L if there exists $y \in Z^d$ and $u_1,..., u_n \in L\mathbb{Z}^d$ such that $\Lambda = y + \bigcup_i Q_L(u_i)$. If f is a function on Ω, Λ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}; is called *local* if Λ_f is finite.

The J_{xy}'s are random variables in an abstract probability space $(\Theta, \mathscr{B}, \mathbb{P})$. $\mathbb{E}(\cdot)$ stands for the expectation with respect to $\mathbb{P}$ (expectation over the disorder). We assume them to be i.i.d. and *bounded*, i.e., there is J_∞ such that $\mathbb{P}\{|J_{xy}| > J_\infty\} = 0$. As a particular case we consider the *diluted Ising model* where J_{xy} is equal to 1 with probability r and equal to 0 with probability $1-r$.

The finite volume Gibbs measure on Λ with boundary condition τ is given by

$$\mu_\Lambda^{J,\tau}(\sigma) = (Z_\Lambda^{J,\tau})^{-1} \exp[-\beta H_\Lambda^{J,\tau}(\sigma_\Lambda)] \tag{1.2}$$

where $Z_\Lambda^{J,\tau}$ is the partition function, and $H_\Lambda^{J,\tau}$ is the finite volume Hamiltonian with b.c. τ. We are always either in the Griffiths phase [G] or in the paramagnetic phase, so there is $\mathbb{P}$-a.s. a unique infinite volume Gibbs measure μ^J. We let $\mu^J(f) = \int \mu(d\sigma) f(\sigma)$.

In this paper we present some new rigorous results on the speed of relaxation to the equilibrium of a Glauber-type dynamics when the system is in the Griffiths regime. Reference [CMM1] contains a gentler introduction to the subject as well as a discussion of previous work. We consider for simplicity the heat-bath dynamics, defined by the transition rates

$$c_J(x, \sigma) = [1 + e^{\beta \nabla_x H_{\{x\}}(\sigma)}]^{-1} \tag{1.3}$$

where $(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$ and $\sigma^x \in \Omega$ is the spin configuration obtained from σ, by flipping the spin at the site x. Any reasonable reversible (i.e., which satisfies the detailed balance condition), attractive single spin-flip dynamics would work. Let $(T_t^{V,J,\tau}; t \geqslant 0)$ denote the semigroup associated with the dynamics in a volume $V \subset \mathbb{Z}^d$, with boundary condition τ. In other words, given a function f of the spin configurations σ, $(T_t^{V,J,\tau} f)(\sigma)$ denotes the expectation of f at time t over the dynamics in V with initial condition σ and boundary condition τ. The quantity T_t^J is the infinite volume semigroup associated with the infinite volume dynamics.

When studying the relaxation to the equilibrium, one is usually interested in either the "worst" initial condition, or the $L^2(\mu^J)$ average, where

μ^J is the equilibrium Gibbs measure. We define then, for a local function f

$$\|T_t^J f - \mu^J(f)\|_\infty = \sup_{\sigma \in \Omega} |(T_t^J f)(\sigma) - \mu^J(f)| \tag{1.4}$$

$$\|T_t^J f - \mu^J(f)\|_{L^2(\mu^J)} = \left(\int \mu^J(d\sigma)\, |(T_t^J f)(\sigma) - \mu^J(f)|^2 \right)^{1/2} \tag{1.5}$$

(remember that μ^J is the unique equilibrium Gibbs measure). The second quantity is a close relative of the usual autocorrelation function. In fact, since T_t^J is self-adjoint on $L^2(\mu^J)$, it follows that

$$\|T_t^J f - \mu^J(f)\|^2_{L^2(\mu^J)} = \langle f(\sigma_0)\, f(\sigma_{2t})\rangle_J - \langle f(\sigma_0)\rangle_J^2$$

where σ_t is the process at time t and $\langle \cdot \rangle$ is the average over the process whose initial condition is thermal equilibrium. For this kind of quantities one can study both

- the *typical* behavior, i.e., properties which have probability one (w.r.t. the disorder), or
- the *average* behavior, which coincides with the average over the spatial translations, since our disorder has an ergodic distribution.

We let Ω_b be the set of all bond configurations, i.e., $\Omega_b = \{0, 1\}^{\mathscr{E}_{\mathbb{Z}^d}}$, where $\mathscr{E}_{\mathbb{Z}^d}$ is the set of all bonds (pairs of nearest neighbors) in $\mathbb{Z}^d$. Then we let P_r be the probability measure on Ω_b associated with an independent bond percolation problem with bond density r and define

$$p_c(d) = \inf \left\{ r : \begin{array}{l} \text{there is a positive } P_r \text{ probability that} \\ \text{the origin belongs to an infinite cluster} \end{array} \right\}$$

If $d \geq 3$, we define $\hat{\beta}(d)$ as the "slab-threshold" for the "pure" (i.e., with all J_{xy} equal to one) Ising model. We refer the reader to [P] for details about $\hat{\beta}_c$ (it is called $\hat{\beta}_l$ over there). The quantity $\hat{\beta}_c(d)$ is conjectured to be equal to the usual critical inverse temperature $\beta_c(d)$. When $d = 2$ we simply let $\hat{\beta}_c(2) = \beta_c(2)$. What is important for us is that for all $\beta > \hat{\beta}_c$ the block magnetization satisfies a surface order large deviation estimate. To be precise let $m_L = (2L+1)^{-d} \sum_{x \in B_L} \sigma(x)$, be the normalized magnetization in B_L, let $m^*(\beta)$ be the spontaneous magnetization (for the pure model), and let $m^{*,f}(\beta) = \lim_{\beta' \to \beta^-} m^*(\beta)$ (i.e., $M^{*,f}$ is the same as m^* unless m^* has a discontinuity). Then, if $\beta > \hat{\beta}_c$, and a, b are such that $-m^{*,f}(\beta) < a < b < m^{*,f}(\beta)$, we have, for large enough L

$$\mu_{B_L}^{\varnothing}\{m_L \in [a, b]\} \leq \exp(-cL^{d-1}) \tag{1.6}$$

where $\mu^{\varnothing}_{B_L}$ is the Gibbs measure in B_L for the pure Ising model with free boundary conditions. Surface order large deviation estimates have been proven in [P] for $d \geqslant 3$. In $d=2$ the problem has been completely solved up to β_c and with an exact expression for c (see [S, CCS, DKS, Pf, I1, I2, CGMS, IS]). The reason why we are interested in inequalities like (1.6) is that they are a key ingredient for proving upper bounds on the relaxation speed (see [CMM1], in particular Theorem 6.4 and the last inequality in the proof of Lemma 6.5).

In the following we will use then FK representation for the Ising model [FK]. We let $\nu_\Lambda^{FK, q, p, w}$ be the *FK* measure in the volume Λ with *wired* b.c. Here q is a positive real number and $p = \{p_{xy}, \langle x, y\rangle \in \mathscr{E}_{\mathbb{Z}^d}\}$ is a collection of real numbers $0 \leqslant p_{xy} \leqslant 1$ each associated to a nearest neighbor bond. The ferromagnetic Ising model (1.1), (1.2) corresponds to $q=2$ and $p_{xy} = 1 - e^{-2\beta J_{xy}}$. We say that there is *wired exponential decay of connectivity* (WEDC) for a given q and p if there exist $C>0, m>0$ such that for all *cubes* Λ we have

$$\nu_\Lambda^{FK, q, p, w}\{x \overset{\Lambda}{\longleftrightarrow} y\} \leqslant C e^{-m|x-y|} \quad \forall x, y \in \Lambda,$$

where $\{x \overset{\Lambda}{\longleftrightarrow} y\}$ is the event that x is connected to y by a path of occupied bonds with both endpoints in $\Lambda \cup \{y\}$. We define $\beta_w = \beta_w(d)$ as the supremum of all β' such that for all $\beta < \beta'$ we have WEDC for $q=2$ and uniform $p_{xy} = 1 - e^{-2\beta}$. We conjecture that $\beta_w = \beta_c$.

We can now state our results:

Theorem 1.1. If $d \geqslant 2$ and either (i) $\mathbb{E}(1 - e^{-2\beta|J_{xy}|}) < p_c(d)$ or (ii) $\mathbb{E}\,|J_{xy}| < \beta_w$, then with $\mathbb{P}$-probability 1 there exists a unique infinite volume Gibbs measure μ^J. Moreover

(a) there exists $k>0$ such that for almost every J and for any local function f there exists $t_0(J, f)$ such that for all $t \geqslant t_0$

$$\|T_t^J f - \mu^J(f)\|_\infty \leqslant \exp[-t \exp[-k(\log t)^{1-1/d} (\log\log t)^{d-1}]] \qquad (1.7)$$

(b) there exists $k>0$ and for any local function f there exists $t_0(f)$ such that, if $t \geqslant t_0(f)$ then

$$\mathbb{E}\,\|T_t^J f - \mu^J(f)\|_\infty \leqslant \exp[-k(\log t)^{d/(d-1)} (\log\log t)^{-d}] \qquad (1.8)$$

The second results shows that the upper bounds are almost saturated in the diluted Ising model. For $x \in \mathbb{Z}^d$ define the function $\pi_x : \Omega \mapsto \{-1, +1\}$, by $\pi_x(\sigma) = \sigma(x)$.

Theorem 1.2. Consider the diluted Ising model with bond density r. Let $d \geqslant 2$, $\beta > \hat{\beta}_c(d)$, and assume that there is a unique Gibbs phase almost surely. Then

(a) if $r > p_c$ there exist $k > 0$ such that, with positive $\mathbb{P}$-probability we have, for t large enough,

$$\|T_t^J \pi_0 - \mu^J(\pi_0)\|_\infty \geqslant \exp[-t \exp[-k(\log t)^{(d-1)/d}]] \tag{1.9}$$

(b) there exists $k > 0$ such that, for large enough t,

$$\mathbb{E}\, \|T_t^J \pi_0 - \mu^J(\pi_0)\|_{L^2(\mu^J)} \geqslant \exp[-k(\log t)^{d/(d-1)}] \tag{1.10}$$

Remark 1. $\mu^J(\pi_0)$ is clearly equal to 0, by uniqueness of the Gibbs measure and the spin-flip symmetry.

Remark 2. Part (b) when $d = 2$ was proven in [CMM2].

Remark 3. Part (a) was proven in [CMM1] for a "uniformly ferromagnetic" models, where J_{xy} is almost surely greater than some constant δ. For diluted systems, the hypothesis $r > p_c$ is crucial. If, in fact, we are below the percolation threshold, then every local function f lives in a finite cluster. By consequence the convergence to equilibrium is exponentially fast with probability 1 and with a J-dependent relaxation time. This implies a *dynamical phase transition* at $r = p_c$.

Remark 4. When $\beta < \beta_c$ the convergence to equilibrium is exponentially fast *uniformly* in the disorder. To see this we can use Theorem 3.1 in [MO] which says that if the interaction has the FKG property and the dynamics is attractive, then, the condition

$$|\mu_{B_L}^{J,+}(\sigma(0)) - \mu_{B_L}^{J,-}(\sigma(0))| \leqslant Ce^{-mL} \tag{1.11}$$

implies $\|T_t^J f - \mu^J(f)\|_\infty \leqslant C'(f)\, e^{-m't}$. Now, thanks to inequality (2.20) in [Hi] we get

$$|\mu_{B_L}^{J,+}(\sigma(0)) - \mu_{B_L}^{J,-}(\sigma(0))| = 2\mu_{B_L}^{J,+}(\sigma(0)) \leqslant \sum_{y \in \Lambda^c} \sum_{\substack{z \in \Lambda \\ |z-y|=1}} \mu_{B_L}^{J,\varnothing}(\sigma(0)\,\sigma(z))$$

By second Griffiths inequality $\mu_{B_L}^{J,\varnothing}(\sigma(0)\,\sigma(z)) \leqslant \mu_{\mathbb{Z}^d}^J(\sigma(0)\,\sigma(z))$. Finally, if $\beta < \beta_c$, the last quantity has an exponential decay in $|z|$ thanks to the results on the sharpness of the phase transition [ABF]. Thus there is also a dynamical phase transition which, modulo the conjecture $\beta_c = \hat{\beta}_c$, occurs exactly at β_c.

Proof of Theorem 1.1. In [CMM1] a general result has been proved which says that the following hypothesis (H) is a sufficient condition for the theorem:

(H) There exist $L_0 \in \mathbb{Z}_+$, $\alpha > 0$, $\vartheta > 0$ such that for all L which are multiples of L_0

$$\mathbb{P}\{SMT(Q_L, L/2, \alpha)\} \geqslant 1 - e^{-\vartheta L}$$

where Q_L is a cube of side L and the *SMT property* is defined as follows: given a finite volume $V \subset \mathbb{Z}^d$, $n \in \mathbb{Z}_+$ and $\alpha > 0$, we say that the condition $SMT(V, n, \alpha)$ holds if for all local functions f and g on Ω such that $d(\Lambda_f, \Lambda_g) \geqslant n$ we have

$$\sup_{\tau \in \Omega} |\mu_V^{J,\tau}(f, g)| \leqslant |\Lambda_f|\ |\Lambda_g|\ \|f\|_\infty\ \|g\|_\infty \exp(-\alpha d(\Lambda_f, \Lambda_g))$$

With $\mu_V^{J,\tau}(f, g)$ we mean the covariance (w.r.t. the measure $\mu_V^{J,\tau}$) of f and g. In [CMM1], (H) is given in an slightly stronger form, i.e., "for all L which are multiples of L_0" is replaced by "for all $L \geqslant L_0$," but it is not difficult to see that the weaker form is sufficient.

So, if we can prove that (H) holds, we are done. To do so we use a result in [CMM2] (Lemma 3.4) which states that (H) is in turn a consequence of the following assumption:

(H′) There exist L_0, $C > 0$ and $m > 0$ such that $SME(\Lambda, C, m)$ holds for *all Λ which are multiples of Q_{L_0}*, where $SME(\Lambda, C, m)$ means that for all $V \subset \Lambda$ and for all $y \in \Lambda^c$, we have

$$\mathbb{E}\{\sup_{\tau \in \Omega} \mathrm{Var}(\mu_{\Lambda, V}^{J,\tau}, \mu_{\Lambda, V}^{J,\tau^y})\} \leqslant C \sum_{x \in V} e^{-m|x-y|}$$

where

$$\mathrm{Var}(\mu_{\Lambda, V}^{J,\tau}, \mu_{\Lambda, V}^{J,\tau^y}) = \max_{\substack{\text{events } F \text{ which depend only} \\ \text{on the spins inside}}} |\mu_\Lambda^{J,\tau}(F) - \mu_\Lambda^{J,\tau^y}(F)| \tag{1.12}$$

and τ^y is the configuration obtained from τ by flipping the spin at y. In order to estimate quantities like (1.12) for general $d \geqslant 2$ we use recent results based on the FK representation [AC]. In particular in [AC] it is shown that

$$\mathrm{Var}(\mu_{\Lambda, V}^{J,\tau}, \mu_{\Lambda, V}^{J,\tau^y}) \leqslant 2\nu_\Lambda^{FK,\, q=2,\, p(J),\, w}(V \overset{\Lambda}{\longleftrightarrow} y) \tag{1.13}$$

where $\nu_{\Lambda}^{FK, q, p, w}$ is the FK measure with a bond-dependent p parameter given by $p_{xy}(J) = 1 - \exp(-2\beta |J_{xy}|)$, and *wired* boundary conditions. $V \overset{\Lambda}{\longleftrightarrow} y$ is the event that V is connected to y by a path of occupied bonds with both endpoints in $\Lambda \cup \{y\}$.

Case (i). $\mathbb{E}(1 - e^{-2\beta |J_{xy}|}) < p_c$. It is well known (see, e.g., [ACCN]) that the FK measure with $q = 1$ dominates (in the FKG sense) the one with $q = 2$ and same p. But $q = 1$ corresponds to independent bond percolation, so, if we let P_p be the probability measure associated with independent bond percolation, we obtain that

$$\nu_{\Lambda}^{FK, 2, p(J), w}\{V \overset{\Lambda}{\longleftrightarrow} y\} \leqslant P_{p(J)}\{V \overset{\Lambda}{\longleftrightarrow} y\} \leqslant P_{p(J)}\{V \overset{\Lambda}{\longleftrightarrow} y\} \quad \forall J \tag{1.14}$$

Moreover it is easy to check that

$$\mathbb{E}P_{p(J)}\{V \leftrightarrow y\} = P_{\bar{p}}\{V \leftrightarrow y\}$$

where $\bar{p} = \mathbb{E}(1 - e^{-2\beta |J_{xy}|})$. In this way we get

$$\mathbb{E}\nu_{\Lambda}^{FK, 2, p(J), w}\{V \overset{\Lambda}{\longleftrightarrow} y\} \leqslant \sum_{x \in V} P_{\bar{p}}\{x \leftrightarrow y\}$$

At this point we use the fact that $\bar{p} < p_c$ and the "absence of intermediate phase" for general percolation models [AB], [MMS] and we get that

$$\sum_{x \in V} P_{\bar{p}}(x \leftrightarrow y) \leqslant \sum_{x \in V} e^{-m |x - y|} \tag{1.15}$$

Case (ii). $\mathbb{E}\,|J_{xy}| < \beta_w$. In this case, thanks to the concavity result, Proposition 1.4, which we prove below, and thanks to the fact that the random variables J_{xy} are independent, we get

$$\mathbb{E}\nu_{\Lambda}^{FK, 2, p(J), w}\{V \overset{\Lambda}{\longleftrightarrow} y\} \leqslant \nu_{\Lambda}^{FK, 2, p(\bar{J}), w}\{V \overset{\Lambda}{\longleftrightarrow} y\} \tag{1.16}$$

where $\bar{J} = \mathbb{E}\,|J_{xy}|$. The result then follows from the definition of β_w.

Proof of Theorem 1.2. We limit ourselves to give a sketch of the proof, since all missing details can be found in the analogous statements in [CMM1] (see, in particular, part (b) of Theorem 3.3, Lemma 6.5 and Lemma 6.6). The only new ingredient is the following result:

Lemma 1.3. Let $d \geqslant 2$ and $r > p_c(d)$. Let $J = \{J_{xy};\ \{x, y\} \in \mathscr{E}_{\mathbb{Z}^d}\}$ be independent Bernoulli random variables with $\mathbb{P}\{J_{xy} = 1\} = r$. Then there exist $\varepsilon > 0$ such that the following holds with positive probability: there exists $\bar{L}(J)$ such that for all $L \geqslant \bar{L}$, the cube Q_L contains a cube $Q_\ell(y)$ of side $\ell = \lfloor(\varepsilon \log L)^{1/d}\rfloor$, such that

(1) $J_{xu} = 1$ if both x and u are in $Q_\ell(y)$

(2) of all bonds $\langle x, u\rangle$ connecting $Q_\ell(y)$ with $Q_\ell(y)^c$, exactly one has $J_{xu} = 1$, while, for all the others we have $J_{xu} = 0$

(3) there is a path of length not greater than L connecting the origin with $Q_\ell(y)$ such that $J_{xu} = 1$ along the path.

Remark. Condition (2) will make our life more complicated, because we are going to deal with an event which is not positive. On the other side having free (or actually "quasi-free") boundary condition is essential in the last inequality in the proof of Lemma 6.5 in [CMM1] where one needs a lower bound on the variance of a certain function (passing to finite volume quantities is understood). That inequality would actually be false in arbitrary b.c. Given Lemma 1.3 (which we prove below), we can complete the argument. The idea is as follows: since $Q_\ell(y)$ is a cube with $\beta > \hat{\beta}_c$ and "quasi-free" b.c. we can use (1.6) and show that the spins $\sigma(u)$ inside $Q_\ell(y)$ relax as (see inequalities (6.34), (6.35) and Lemma 6.5 in [CMM1])

$$\|T_t^J \pi_u\|_\infty \geqslant \frac{c_1}{\ell^d} \exp[-te^{-c_2\ell^{d-1}}] \geqslant \frac{c_1}{\varepsilon \log L} \exp[-te^{-c_3(\log L)^{(d-1)/d}}] \qquad (1.17)$$

So we know that for all L large enough, within distance L from the origin there are spins relaxing as slow as in (1.17). At first sight, this doesn't seem to say much about the spin at the origin yet. But it is not too difficult to show that this slow relaxation actually propagates all the way to the origin, in some weaker form, if (of course) the origin is connected to the slow relaxing spins. More precisely one can prove (see (6.33) in [CMM1]) that, if 0 and u are connected by a path of length n then

$$\|T_t^J \pi_0\|_\infty \geqslant \|T_t^J \pi_u\|_\infty e^{-mn} \qquad (1.18)$$

for some m independent of J. From (1.17) and (1.18) we get

$$\|T_t^J \pi_u\|_\infty \geqslant \frac{c_1}{\varepsilon \log L} \exp[-mC(r)L - te^{-c_3(\log L)^{(d-1)/d}}]$$

The result follows by choosing $L \sim t \exp[-(\log t)^{(d-1)/d}]$ and from the fact that $\mu^J(\pi_0) = 0$. ∎

Proof of Part (b). This statement has been proven in [CMM1] for β larger than some unspecified $\beta_1(d)$. The proof can be extended to all

$\beta > \hat{\beta}_c$ in exactly the same way as it was done for the two-dimensional case (Proposition 4.3 in [CMM2]). ▮

Proof of Lemma 1.3. Assume first $d > 2$. Consider the event $\mathscr{E}$ that in Q_L there is a cube $Q_\ell(y)$ with $\ell = \lfloor (\varepsilon \log L)^{1/d} \rfloor$, such that (1), (2) and (3) of Lemma 1.3 hold.

We use the fact [GMa] that if we are beyond the percolation threshold, for b sufficiently large, there is percolation in the slab of thickness b; i.e., the region

$$\mathscr{Q} = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0, 1,..., b-1\}^{d-2}.$$

Let then $\mathscr{Q}_L = \{0,..., L-1\}^2 \times \{0, 1,..., b-1\}^{d-2}$, be the two dimensional L by L square thickened by b in the other $d-2$ directions. We define $\mathscr{P}_2\colon \mathbb{Z}^d \mapsto \mathbb{Z}^2$ as the projection on the first 2 coordinates. Let X_∞ be the event that the cluster of the origin constructed using the occupied bonds in $\mathscr{Q}$ is infinite, and choose b such that $P_r(X_\infty) > 0$. On the event X_∞ we can define random sites (see Fig. 1) $u_n \in \mathscr{Q}$ for $n = 1, 2,...$ such that

(1) $|u_n| = 2n\ell$ a.s.

(2) u_n is connected to the origin by a path of occupied bonds inside $\mathscr{Q}_{2n\ell}$.

If there is more than one site satisfying (1) and (2), pick one of them with a given arbitrary rule. Let then v_n be a nearest neighbor of u_n obtained by adding 1 to either the first or the second coordinate of u_n, in such a way that $|v_n| = 2n\ell + 1$. So v_n is still inside $\mathscr{Q}$ but just outside $\mathscr{Q}_{2n\ell}$. What we would like to say is that there is now a cube $\mathscr{Q}_\ell$ at v_n which has some finite

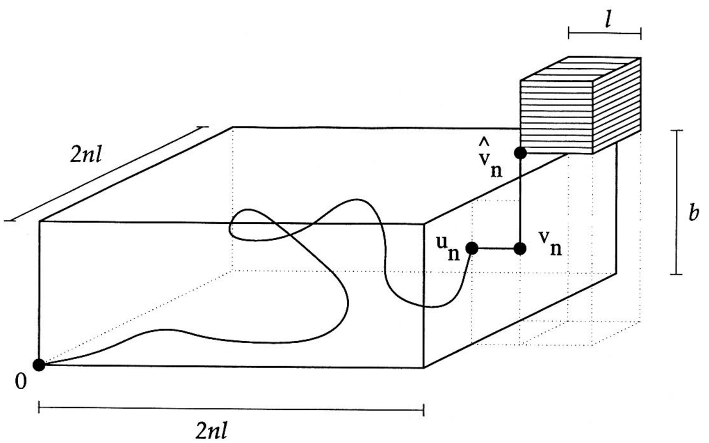

Fig. 1. Proof of Lemma 1.3 when $d > 2$.

probability to satisfy statements (1) and (2) of the Lemma. Unfortunately statement (2) does not translate into a positive event, so we are going to push the cube Q_ℓ out of the quadrant slab, along, say the third direction and construct, by hand, a connection to it. We denote by $x^{(i)}$ the ith coordinate of some $x \in \mathbb{Z}^d$. Thus we let $\hat{v}_n = (v_n^{(1)}, v_n^{(2)}, b, 0,..., 0)$ and let $c_n = Q_\ell + \hat{v}_n$, so that c_n is right "above" the slab. Consider then the events

(1) F_n that the bond $\{u_n, v_n\}$ and the bonds "between" v_n and $\hat{v}_n$ are occupied,

(2) $c_n^\#$ that all bonds inside c_n are occupied, while all bonds connecting c_n with $(c_n)^c$ are empty, except the bond $\{\hat{v}_n, \hat{v}_n - (0, 0, 1, 0,..., 0)\}$.

Let then $G_n = F_n \cap c_n^\#$. We observe that the occurrence of one of the G_n's for some $n \leqslant \lfloor L^{1/(2d)} \rfloor$ implies $\mathscr{E}$, because the length of the path connecting c_n to the origin is at most the number of bonds in $\mathscr{Q}_{2n\ell}$ plus $b+1$, and so it is not greater than L. Thus we let $N = \lfloor L^{1/(2d)} \rfloor$, and we want an upper bound on the probability that none of the G_n, $n \leqslant N$ occur. Let then $\mathscr{G}_k = \bigcup_{n=1}^k G_n$. We have

$$P_r(\mathscr{G}_N^c \mid X_\infty) = P_r(G_N^c \mid \mathscr{G}_{N-1}^c, X_\infty)\, P_r(\mathscr{G}_{N-1}^c \mid X_\infty) \tag{1.19}$$

We argue that

$$P_r(G_N^c \mid \mathscr{G}_{N-1}^c, X_\infty) \leqslant P_r(G_N^c). \tag{1.20}$$

In fact, since $G_N = F_N \cap c_N^\#$, and since $c_N^\#$ is independent of F_N, $\mathscr{G}_{N-1}$ and X_∞, we have that

$$P_r(G_N \mid \mathscr{G}_{N-1}^c, X_\infty) = P_r(c_N^\#)\, P_r(F_N \mid \mathscr{G}_{N-1}^c, X_\infty) \tag{1.21}$$

Let now $\hat{\mathscr{Q}}_n = \{0,..., n-1\}^2 \times \mathbb{Z}^{d-2}$ and let $\mathscr{B}_n$ be the sigma algebra generated by all the bond variables with both endpoints in $\hat{\mathscr{Q}}_n$. Then we have $\mathscr{G}_{N-1}^c \in \mathscr{B}_{2(N-1)\ell+\ell+2}$. Thus we can write

$$P_r(F_N \mid \mathscr{G}_{N-1}^c, X_\infty) = \frac{\int_{\mathscr{G}_{N-1}^c} P_r(d\omega)\, P_r(F_N \cap X_\infty \mid \mathscr{B}_{2(N-1)\ell+\ell+2})(\omega)}{P_r(X_\infty \cap \mathscr{G}_{N-1}^c)} \tag{1.22}$$

Now we use the fact that the conditional probability $P_r(\cdot \mid \mathscr{B}_n)$ has still the FKG property, which together with the fact that F_N and X_∞ are positive events and that F_N is independent of $\mathscr{B}_{2(N-1)\ell+\ell+2}$ yields

$$P_r(F_N \cap X_\infty \mid \mathscr{B}_{2(N-1)\ell+\ell+2}) \geqslant P_r(F_N)\, P_r(X_\infty \mid \mathscr{B}_{2(N-1)\ell+\ell+2}) \tag{1.23}$$

From (1.21), (1.22), (1.23) we get $P_r(G_N \mid \mathscr{G}^c_{N-1}, X_\infty) \geqslant P_r(F_N)\, P_r(c^\#_N) = P_r(G_N)$, so (1.20) is proven. Moreover, if we let δc_N be the set of all bonds connecting c_N with c^c_N, we get

$$P_r(G_N) \geqslant r^{b+1} r^{d\,|c_N|}(1-r)^{|\delta c_N|} \geqslant e^{-K\varepsilon \log L} \tag{1.24}$$

for some $K = K(r, b)$, if L is large enough. From (1.19), (1.20) and (1.24), we obtain

$$P_r\left(\bigcap_{n=1}^{N} G^c_n \,\middle|\, X_\infty\right) \leqslant (1 - e^{-K\varepsilon \log L})\, P_r\left(\bigcap_{n=1}^{N-1} G^c_n \,\middle|\, X_\infty\right)$$

Iterating

$$P_r\left(\bigcap_{n=1}^{N} G^c_N \,\middle|\, X_\infty\right) \leqslant (1 - e^{-K\varepsilon \log L})^N$$

Since $N = \lfloor L^{1/(2d)} \rfloor$, if ε is small enough, we have that

$$P_r\left(\bigcup_{n=1}^{N} G_N \,\middle|\, X_\infty\right) \geqslant 1 - \exp[-L^{1/(3d)}]$$

By the Borel–Cantelli lemma, we thus obtain that with $P_r(\cdot \mid X_\infty)$-probability 1, there exists L_0 such that $\mathscr{E}$ holds for all $L \geqslant L_0$.

The $d = 2$ Case. If $d = 2$ there is no way of pushing the cube c_n "above" the quadrant, so we need a different approach. We let X_∞ be the event that the origin percolates within the quadrant $\mathbb{Z}^+ \times \mathbb{Z}^+$, and use again the fact that $P_r(X_\infty) > 0$ if r is above the percolation threshold. Let $\ell_1 = \lfloor L^{1/4} \rfloor$.

$$C_k = W_{\ell_1} + 0_k \qquad \text{where} \quad 0_k = -(k\ell_1, 0, \ldots, 0)$$

In particular (see Fig. 2) we want to focus on $C_0, C_1, \ldots, C_{\ell_1}$.

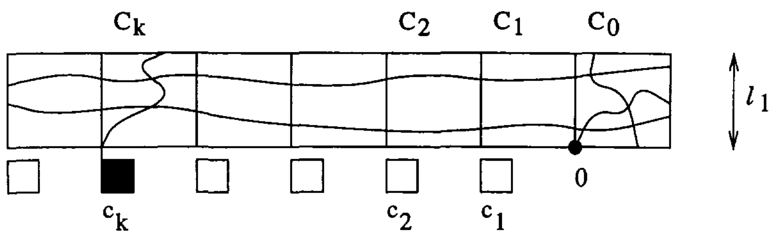

Fig. 2. Proof of Lemma 1.3 when $d = 2$.

We also let

$$c_k = Q_\ell - (k\ell_1, \ell + 1)$$

where $\ell = \lfloor \varepsilon \log L \rfloor$, with ε to be determined momentarily. Finally we define H to be the horizontal "strip" given by $H = \bigcup_{k=0}^{\ell_1} C_k$. For any of the objects defined above, we use the superscripts T, B, L, R to denote its top, bottom, left and right side. Consider then the events

$$\{A \overset{C}{\longleftrightarrow} B\} = \{A \text{ is connected to } B \text{ inside } C\}$$

$$c_k^\# = \left\{ \begin{array}{c} \text{all bounds in } c_k \text{ are occupied, all bonds connecting } c_k \\ \text{with } (c_k)^c \text{ are empty except the bond connecting} \\ \text{the top left corner of } c_k \text{ to the bottom left corner} \\ \text{of } C_k \text{ which is occupied} \end{array} \right\}$$

$$G_k = \{0_k \overset{C_k}{\longleftrightarrow} C_k^T\} \cap c_k^\#$$

and define $\mathscr{G} = \bigcup_{k=1}^{\ell_1} G_k$. Finally, let

$$E = \{H^L \overset{H}{\longleftrightarrow} H^R\} \cap \{C_0^B \overset{C_0}{\longleftrightarrow} C_0^T\} \cap \mathscr{G}$$

It is easy to see that on X_∞, the event E implies the event $\mathscr{E}$. In fact if X_∞ and E occur, then there is a path from the origin to c_k whose length cannot exceed the number of bonds in H, and so it is not greater than $2\ell_1{}^3 \leqslant L$. On the other side we have

$$P_r(E^c \mid X_\infty) \leqslant P_r(\{H^L \overset{H}{\longleftrightarrow} H^R\}^c \mid X_\infty) + P_r(\{C_0^B \overset{C_0}{\longleftrightarrow} C_0^T\}^c \mid X_\infty) + P_r(\mathscr{G}^c \mid X_\infty) \tag{1.25}$$

In the first two terms there is a positive event conditioned to another positive event, so we use the FKG inequality and we get an upper bound by removing the conditioning. In the last term we observe that $\mathscr{G}$ is independent of X_∞. So we get

$$P_r(E^c \mid X_\infty) \leqslant P_r(\{H^L \overset{H}{\longleftrightarrow} H^R\}^c) + P_r(\{C_0^B \overset{C_0}{\longleftrightarrow} C_0^T\}^v) + P_r(\mathscr{G}^c) \tag{1.26}$$

Since we are above the percolation threshold, the first two terms are not greater than $1 - c_1 e^{-m_1 L^{1/4}}$ for some constants $c_1 < \infty$ and $m_1 > 0$.

As for the third term in (1.26) we notice that G_k and $G_{k'}$ are independent if k, k' are two distinct odd integers. Moreover $\{0_k \overset{C_k}{\longleftrightarrow} C_k^T\}$ and $c_k^\#$ are also independent. Finally, we are above the (quadrant) percolation threshold, so $P_r\{0_k \overset{C_k}{\longleftrightarrow} C_k^T\} > \vartheta$ for some positive ϑ independent of k. Thus, for large L

$$P_r(\mathcal{G}^c) \leqslant \prod_{k \text{ odd}} [1 - P_r(G_k)] \leqslant (1 - \vartheta r^{3\varepsilon \log L})^{\ell_1/2} \leqslant e^{-L^{1/8}}$$

if $3\varepsilon\, |\log r| \leqslant 1/10$ and L is large enough. So we have

$$P_r(\mathscr{E} \mid X_\infty) \geqslant P_r(E \mid X_\infty) \geqslant 1 - 3\exp[-L^{1/8}]$$

and we conclude as in the $d > 2$ case, by using the Borel–Cantelli lemma. ∎

We finish by proving the "concavity property" used in the inequality (1.16) which can be thought of as a generalization of a result in [OPG]. We define $\bar{\mathscr{E}}$ as the set of all bonds of $\mathbb{Z}^d$ with at least one endpoint in Λ.

Proposition 1.4. Let Λ be a finite subset of $\mathbb{Z}^d$, f be a non-decreasing function on $\{0, 1\}^{\bar{\mathscr{E}}_\Lambda}$. Let $J = \{J_{xy}, \langle x, y\rangle \in \mathscr{E}_{\mathbb{Z}^d}\}$ be a collection of positive real numbers and let $p = \{p_{xy}, \langle x, y\rangle \in \mathscr{E}_{\mathbb{Z}^d}\}$ where $p_{xy} = 1 - e^{-2J_{xy}}$. Consider the associated FK probability $\nu_\Lambda^{FK, 2, p(J), w}$. Then $\nu_\Lambda^{FK, 2, p(J), w}(f)$ is concave in each J_{xy}.

Proof. Throughout this proof we let, for simplicity, $\nu^{q, p} = \nu_\Lambda^{FK, q, p, w}$. Choose a particular bond $e = \langle x, y\rangle$ and let

$$F(p_e) = \nu^{q, p}(f) \qquad \text{and} \qquad G(J_e) = F(1 - e^{-2J_e})$$

We want to prove that $G'' \leqslant 0$ if $q = 2$. In fact we have

$$F(p_e) = Z(p_e)^{-1} \sum_{n \in \{0, 1\}^{\bar{\mathscr{E}}_\Lambda}} h(n)\, f(n)\, p_e^{n_e}(1 - p_e)^{1 - n_e} \tag{1.27}$$

where Z is the partition function for the FK-model and

$$h(n) = q^{\#\ \text{clusters}} \prod_{b \neq e} p_b^{n_b}(1 - p_b)^{1 - n_b}$$

If we now denote by g the function of the variable n_e

$$g_{p_e}(n_e) = \frac{(2n_e - 1)}{p_e^{n_e}(1 - p_e)^{1 - n_e}} \tag{1.28}$$

we get that

$$\begin{aligned} F'(p_e) &= \nu^{q,p}(fg_{p_e}) - \nu^{q,p}(f)\, \nu^{q,p}(g_{p_e}) \\ F''(p_e) &= -2F'(p_e)\, \nu^{q,p}(g_{p_e}) \end{aligned} \tag{1.29}$$

which implies that

$$\begin{aligned} G''(J_e) &= 2\exp(-2J_e)[F''e^{-2J_e} - 2F'] \\ &= -4\exp(-2J_e)\, F'(p_e)[1 + 2e^{-2J_e}\nu^{q,p}(g_{p_e})] \end{aligned}$$

Notice that F' is non negative because both f and g_{p_e} are increasing functions and the measure $\nu^{q,p}$ satisfies the FKG property. Thus, in order to conclude the proof, we have to show that $1 + 2e^{-2J_e}\nu^{q,p}(g_{p_e}) \geqslant 0$. By explicit computation we have

$$\begin{aligned} 1 + 2e^{-2J_e}\nu^{q,p}(g_{p_e}) &= 1 + 2(1 - p_e)\left[\frac{\nu^{q,p}(n_e = 1)}{p_e} - \frac{\nu^{q,p}(n_e = 0)}{(1 - p_e)}\right] \\ &= 2\left[\frac{\nu^{q,p}(n_e = 1)}{p_e} - \frac{1}{2}\right] \end{aligned}$$

Finally we observe that the FK-model with parameters (q, p) dominates an independent percolation model with parameter $p' = p/[q - (q-1)\,p]$ (see, e.g., [ACCN]). Thus, for $q = 2$ we get

$$\frac{\nu^{q,p}(n_e = 1)}{p_e} - \frac{1}{2} \geqslant \frac{1}{2 - p_e} - \frac{1}{2} \geqslant 0 \quad \blacksquare$$

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