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MIXED PERCOLATION AS A BRIDGE BETWEEN SITE AND BOND PERCOLATION

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By using mixed percolation as a bridge between site and bond percolation, we derive a new inequality between the critical points of these processes that is optimal in a certain sense. We also extend a result on the crossover exponent of bond-diluted Potts models to site-diluted Potts models. Some new results about the critical line in mixed percolation are also proved.

1. Introduction. Bond and site percolation are similar processes in various respects, and it is easy to overlook their differences and miss the fact that their comparison may sometimes be interesting and challenging. The issues addressed in this paper include a nontrivial relationship between the critical points of these two processes and a nontrivial extension to the site percolation setting of a result on diluted Potts models that was only available in the bond percolation setting. For these purposes, we study mixed site–bond percolation, which can be seen as a smooth bridge between site and bond percolation. As a by-product of our approach, we also derive some new results on the critical line in mixed percolation.

1.1. *Comparison between critical points.* We consider site or bond percolation on an infinite, locally finite, connected graph $G = (V, E)$. Here V is the set of vertices (sites) and E is the set of edges (bonds). The degree of a vertex is the number of edges incident to it. If the degree of each site is bounded above by a common finite constant, the graph G is said to be of bounded degree. Write $\mathrm{Aut}(G)$ for the group of graph automorphisms of G, that is, the class of one-to-one transformations of V onto itself that preserve the graph structure. A graph $G = (V, E)$ is called transitive if for any $x, y \in V$ there exists a graph automorphism of G that maps x to y. In other words, G is transitive if $\mathrm{Aut}(G)$ acts transitively on V, that is, it produces a single orbit. Intuitively, a graph is transitive if all its vertices play exactly the same role. A graph $G = (V, E)$ is called quasitransitive if $\mathrm{Aut}(G)$ acting on V produces a finite number of orbits. Intuitively, a graph is quasitransitive if it has a finite number of types of vertices and vertices of the same type play exactly the same role.

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In the site percolation model, each site is kept independently of the others with probability s, while the other sites are removed along with all the bonds incident to them. We denote by $\pi^S(s)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). In the bond percolation model, each bond is kept independently of the others with probability b, while the other bonds are removed. We denote by $\mathrm{B}(b)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). The critical points are defined by

$$s_c = \inf\{s \in [0,1] : \pi^{\mathrm{S}}(s) > 0\} = \inf\{s \in [0,1] : \pi^{\mathrm{S}}(s) = 1\},$$
$$b_c = \inf\{b \in [0,1] : B(b) > 0\} = \inf\{b \in [0,1] : B(b) = 1\},$$

where the second equality in each definition is a consequence of Kolmogorov's 0–1 law.

The inequality

$$b_c \le s_c, \tag{1}$$

which saturates for trees, is owing to [14]. In particular, if $s_c < 1$, then $b_c < 1$. In contrast, there are locally finite graphs (of unbounded degree) for which $b_c < 1$, while $s_c = 1$. For this, see [12]. For graphs of bounded degree, the following inequality, from [11], shows that this cannot happen:

$$s_c \le 1 - (1 - b_c)^{D-1}, \tag{2}$$

where D is the maximal degree of the graph. [Relationships that are weaker than (2), but sufficient to show that for bounded degree graphs, $b_c < 1$ implies $s_c < 1$, can be obtained from [17] and [18], Remark 6.2; see also [12].]

In [11], the authors implicitly asked if (2) is optimal or not. Theorem 1.1 shows that this is not the case when b_c is close to 1 and provides an inequality that is optimal "up to the value of a constant prefactor." Recall that $[x]$ denotes the integer part of x.

THEOREM 1.1. *For each integer $D \ge 3$, the following statements hold.*

(a) *For any infinite connected graph of maximal degree D,*

$$s_c \le 1 - \frac{1}{2^{2\lfloor D/2\rfloor+1}}(1 - b_c)^{\lfloor D/2\rfloor}.$$

(b) *There are infinite connected quasitransitive graphs of maximal degree D for which $b_c < 1$ and*

$$s_c \ge 1 - 2^{\lfloor D/2\rfloor+1}(1 - b_c)^{\lfloor D/2\rfloor};$$

examples of such graphs can be chosen with b_c arbitrarily close to 1.

Note that an infinite graph of maximal degree at most 2 has $b_c = s_c = 1$, so the assumption that $D \ge 3$ is harmless. The optimal aspect of Theorem 1.1 is the identification of the exponent $\lfloor D/2\rfloor$, which is the same in parts (a) and (b). The value of the constant that multiplies $(1 - b_c)^{\lfloor D/2\rfloor}$ in parts (a) and

(b) can be slightly improved using our techniques with somewhat more work. However, because these techniques do not seem to be sufficient to identify the optimal value of this constant, we satisfy ourselves with the preceding statements.

OPEN PROBLEM 1. How would Theorem 1.1 be modified if we restricted ourselves to the class of transitive graphs of degree D?

1.2. *Crossover exponent for diluted Potts models.* We suppose that the reader is familiar with the basic notions and terminology on the statistical mechanics of lattice systems. For an introduction to this subject, consult, for example, [7]. The q-state Potts model on a locally finite (not necessarily infinite or connected) graph $G = (V, E)$ is defined by associating to each site $x \in V$ a random variable (spin) $\sigma(x)$ that takes values $1, 2, \dots, q$. To each configuration $\sigma \in \{1, 2, \dots, q\}^V$, an energy is formally associated by the expression

$$H_J(\sigma) = -J \sum_{\{x, y\} \in E} \delta_{\sigma(x), \sigma(y)},$$

where $J \geq 0$ is a parameter and $\delta_{a, b} = 1$ if $a = b$, $\delta_{a, b} = 0$ if $a \neq b$. There is a critical value J_c (possibly 0 or ∞) such that if $J < J_c$, the model has a unique Gibbs distribution, but if $J > J_c$, the model has more than one Gibbs distribution. (If G is not connected, a Gibbs distribution on it is obtained by associating an independent Gibbs distribution to each connected component, so that the value of J_c is then the infimum of those of its connected components.) For an introduction to the Potts model and the associated random-cluster models, see, for example, [8]. It is of relevance to recall that the case $q = 2$ corresponds to the Ising model.

Given an infinite, locally finite, connected graph $G = (V, E)$, we can consider the bond-diluted and the site-diluted q-state Potts models on this graph. In each case, we study the q-state Potts model on the random graph resulting from performing bond or site percolation on G. (Note that this resulting graph is, in general, not connected, but as observed in the last paragraph, this is not an important issue.) The study of statistical mechanics models on diluted graphs is a chapter in the study of systems in random environment, a very central area of research in physics and in mathematical physics. For some of the background on the subject, consult [1] and [13] and references therein.

We can show, using Kolmogorov's 0–1 law and Theorem 7.33 of [7] or Theorem 3.1 of [4], that for each graph the critical value $J_c^B(b)$ [resp. $J_c^S(s)$] for the q-state Potts model on the corresponding bond-diluted (resp. site-diluted) graph is well defined and not random.

In [1], the following theorem was proved [see display (9) and (10) in that paper].

THEOREM 1.2. *For each infinite, locally finite, connected graph G that has $0 < b_c < 1$ and for each q, there exist constants $0 < A_1^B \leq A_2^B < \infty$ such that* *for the bond-diluted q-state Potts model on G.*

$$A_1^B(b-b_c) \le \exp(-J_c^B(b)) \le A_2^B(b-b_c)$$

for $b > b_c$. Given $\delta > 0$, we can take $A_1^B = (1-\delta)/(qb_c)$ and $A_2^B = 1/b_c$, provided that we restrict ourselves to values of b close enough to b_c.

In particular, the crossover exponent Φ^B, defined by the relationship

$$\exp(-J_c^B(b)) \sim (b-b_c)^{\Phi^B} \quad \text{in the logarithmic sense, as } b \searrow b_c,$$

takes the value $\Phi^B = 1$.

The methods in [1] are based on the relationship the Potts model and the randomcluster model (a certain dependent bond percolation model) and on comparisons between the random-cluster model and independent bond percolation. The method does not extend to site-diluted Potts models, because the random-cluster model is not immediately comparable to independent percolation. Using the techniques from the current paper, we can fill this gap and prove the following theorem.

THEOREM 1.3. *For each infinite connected graph of bounded degree G that has $b_c < 1$ and for each q, there exist constants $0 < A_1^S \le A_2^S < \infty$ that depend only on q and D, such that for the site-diluted q-state Potts models on G,*

$$A_1^S(s-s_c) \le \exp(-J_c^S(s)) \le A_2^S(s-s_c)$$

for $s > s_c$. Given $\delta > 0$, we can take $A_1^S = (1-\delta)/(qs_c)$ and $A_2^S = (1+\delta)2^D/(s_c(1-s_c)^{D-2})$, where D is the maximal degree of G, provided that we restrict ourselves to values of s close enough to s_c.

In particular, the crossover exponent Φ^S, defined by the relationship

$$\exp(-J_c^S(s)) \sim (s-s_c)^{\Phi^S} \quad \text{in the logarithmic sense, as } s \searrow s_c,$$

takes the value $\Phi^S = 1$.

As far as we can tell, the comparisons between independent bond and site percolation available in the literature (see [17], [18], Remark 6.2, and [11]), when combined with the arguments from [1], allow us to prove only a weaker result, which would only imply that if Φ^S exists, then it is positive and finite. Results of this type are contained in [5] and [6].

Regarding the motivation behind the study of systems in a random environment, it seems natural to consider site-diluted Potts models as more interesting than bond-diluted models, because site dilution corresponds to the presence of impurities in a crystal. We stress also that although Theorem 1.3 is stated in the context of fairly arbitrary graphs, as far as we know, it is a new result even in the special case of cubic lattices.

1.3. *Mixed percolation.* In the mixed percolation model, each bond is kept independently of anything else with probability b and each site is kept independently of anything else with probability s; the bonds and sites that are not kept as well as the bonds incident to these sites are removed. We denote by $\pi^{\mathrm{BS}}(b, s)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). It is natural to define, for $0 \le b \le 1$,

$$s_c(b) = \sup\{s \in [0, 1] : \pi^{\mathrm{BS}}(b, s) = 0\}.$$

THEOREM 1.4. *For any infinite connected graph of bounded degree that has* $b_c < 1$, *the function* $s_c(\cdot)$ *has the following properties*:

(a) *There exist* $0 < C_1 \le C_2 < \infty$, *which depend only on the maximal degree of the graph*, *such that for any* $b', b'' \in [b_c, 1]$,

$$C_1|b' - b''| \le |s_c(b') - s_c(b'')| \le C_2|b' - b''|.$$

(b) $s_c(b_c) = 1$.

In particular, Theorem 1.4 implies that $s_c(\cdot)$ is a Lipschitz continuous function on [0,1], which is identically 1 on $[0, b_c]$ and strictly decreasing on $[b_c, 1]$.

It is also natural to define

$$b_c(s) = \inf\{b \in [0, 1] : \pi^{\mathrm{BS}}(b, s) > 0\} = \inf\{b \in [0, 1] : \pi^{\mathrm{BS}}(b, s) = 1\}.$$

Theorem 1.4 can be rephrased in terms of the function $b_c(\cdot)$. Note also that from this theorem, we obtain that the function $s_c(\cdot)$ restricted to the domain $[b_c, 1]$ is the inverse of the function $b_c(\cdot)$ restricted to the domain $[s_c, 1]$. Moreover, the boundary between the percolative and the nonpercolative regions in the plane (b, s) is precisely the curve

$$\{(b, s_c(b)) : b_c \le b \le 1\} = \{(b_c, (s), s) : s_c \le s \le 1\}.$$

An important piece of information contained in Theorem 1.4 refers to the edges of this curve. It implies that for $b_c \le b \le 1$,

$$C_1(b - b_c) \le 1 - s_c(b) \le C_2(b - b_c) \tag{3}$$

and

$$C_1(1 - b) \le s_c(b) - s_c \le C_2(1 - b). \tag{4}$$

In particular, the critical exponents defined by

$$1 - s_c(b) \sim (b_c - b)^{\Phi'} \quad \text{in the logarithmic sense, as } b \searrow b_c$$

and

$$s_c(b) - s_c \sim (1 - b)^{\Phi''} \quad \text{in the logarithmic sense, as } b \nearrow 1,$$

take the value $\Phi' = \Phi'' = 1$. The relationships in (4) allow us to prove Theorem 1.3.

OPEN PROBLEM 2. How smooth is the function $s_c(\cdot)$ with domain restricted to $[b_c, 1]$? It is not hard to find graphs that are not quasitransitive for which this function is not differentiable at some points and also has inflection points. However, it is not clear whether such behaviour could happen on a quasitransitive graph. In particular, it is natural to conjecture (based on simulations and exact results for homogeneous trees) that in the case of the cubic lattice $\mathbb{Z}^d$, $d \geq 2$, this function is analytic and convex.

In Section 2 we derive a differential inequality that is used in Section 3 to derive estimates on the function $s_c(\cdot)$, and so provide the proof of Theorem 1.4. In Section 4, these estimates on $s_c(\cdot)$ are used to prove part (a) of Theorem 1.1; the example that proves part (b) of this theorem also is presented there. Finally, in Section 5 the proof of Theorem 1.3, based on (4) is presented.

2. Differential inequality. In this section we consider mixed percolation on an infinite connected graph G of bounded degree. A chain is a finite sequence of distinct sites, $x_0, \dots, x_n$, such that for $i = 0, \dots, n-1$, x_i and x_{i+1} are joined by an edge. These edges are called the edges of the chain. The length of such a chain is n, and its end points are x_0 and x_n. The distance between two sites is the minimal length of the chains that have these sites as end points. An arbitrary site of the graph is chosen as its root and denoted by r. The ball of radius n centered at the root, denoted $V(n)$, is the set of sites within distance n of the root. We also set $\widetilde{V}(n) = V(n) \backslash \{r\}$.

Clusters are the connected components of the graph obtained from G after sites and bonds have been deleted in the percolation process. We denote by $\theta^{\mathrm{BS}}(b, s)$ the probability that for mixed percolation with parameters (b, s) on G, the root belongs to an infinite cluster. It is well known and easy to see that $\pi^{\mathrm{BS}}(b, s) = 1$ iff $\theta^{\mathrm{BS}}(b, s) > 0$ and that, in particular, this last statement is independent of the choice of the root of G.

As usual, bonds and sites that are kept in the percolation process are said to be occupied and those that are removed are said to be vacant.

A chain is said to be internally occupied if all its sites, except possibly for its end points, and all its edges are occupied. Two sites are said to be almost connected if there is an internally occupied chain that has these sites as end points. We denote by A_n the event that the root is almost connected to a site that is at distance $n+1$ from the root. Set $\tilde{\theta}_n^{\mathrm{BS}}(b, s) = P(A_n)$. Denote, as usual, by $\theta_n^{\mathrm{BS}}(b, s)$ the probability that A_n happens and the root also is occupied. Then it is clear that

$$\lim_{n\to\infty} s\tilde{\theta}_n^{\mathrm{BS}}(b, s) = \lim_{n\to\infty} \theta_n^{\mathrm{BS}}(b, s) = \theta_n^{\mathrm{BS}}(b, s);$$

in particular,

$$\pi^{\mathrm{BS}}(b, s) = 1 \quad \text{iff} \quad \tilde{\theta}^{\mathrm{BS}}(b, s) = \lim_{n\to\infty} \tilde{\theta}_n^{\mathrm{BS}}(b, s) > 0. \tag{5}$$

PROPOSITION 2.1. *For any infinity connected graph of degree bounded above by $D < \infty$, the differential inequality*

$$\frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial s} \le \widetilde{C}(b,s) \frac{\tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial b} \tag{6}$$

holds, where $\widetilde{C}(b,s)$ *can be chosen as* $2^{\lfloor D/2 \rfloor}/(s(1-b)^{\lfloor D/2 \rfloor - 1})$ *or as* $2^D/(s(1-s)^{D-2})$.

Under the same assumptions, we can also prove a differential inequality of the form

$$\frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial s} \le C(b,s) \frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial b}. \tag{7}$$

However, the corresponding estimate on $C(b,s)$ turns out to be not good enough to prove Theorem 1.1(a) (it is good enough, though, to prove Theorems 1.4 and 1.3).

Note that two alternative expressions for $\widetilde{C}(b,s)$ appear in Proposition 2.1. The first alternative is important in the proof of Theorem 1.1(a) (the power of $1-b$ that appears in this expression is of major relevance then); the second is important in the proof of Theorem 1.3 (because it does not blow up as b approaches 1). Both expressions are needed to prove Theorem 1.4 [because they allow us to bound $\widetilde{C}(b,s)$ above, uniformly on a domain that excludes a neighborhood of the point $(b,s) = (1,1)$ and a neighborhood of the line $s = 0$].

Under the same assumptions of Proposition 2.1, we can also prove the complementary differential inequalities

$$\frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial b} \le \frac{D}{b} \frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial s} \tag{8}$$

and

$$\frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial b} \le \frac{D}{b} \frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial s}. \tag{9}$$

All the applications that we would have for them are, nevertheless, better accomplished by using (1).

Differential inequalities in the spirit of (6), (7), (8) and (9) have appeared in [19], [2], [3] and [10]. They have been used to prove strict inequalities between critical points, equalities between critical exponents obtained by approaching a critical point from different directions and Lipschitz continuity of critical lines. The main applications in this paper are, nevertheless, of a different nature. In particular, as far as we know, in previous uses of such inequalities, estimates on the values of the constants that played the role of our $\widetilde{C}(b,s)$ were not a crucial issue.

PROOF OF PROPOSITION 2.1. Recall the notation introduced in the beginning of this section. We also use the notation E_x to denote the set of edges incident to the vertex x and use V_x to denote the set of vertices that share an edge with x. Also let $E(n)$ be the set of edges that have at least one end point in the ball $V(n)$. To prove the proposition, we have to consider only mixed percolation on the finite graph $G_n = (V(n), E(n))$. A realization of this process is a function ω: $V(n) \cup E(n) \to \{0, 1\}$, where the 0's associated to vacant bonds and sites, and the 1's are associated to occupied bonds and sites.

Given a site $x \in \widetilde{V}(n)$, let $\delta_x A_n$ be the event that x is pivotal for A_n, that is, the event that if x is occupied, then A_n happens, but if x is vacant, then A_n does not happen. Given a bond $e \in E(n)$, the event $\delta_e A_n$ that e is defined in an analogous way.

Russo's formula (see, e.g., [9]), states that

$$\frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial s} = \sum_{x \in \widetilde{V}(n)} P(\delta_x A_n), \qquad \frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial b} = \sum_{e \in E(n)} P(\delta_e A_n).$$

The proof is, therefore, concluded once we argue that for each $x \in \widetilde{V}(n)$,

$$sP(\delta_x A_n) \le \left(\frac{2}{1-b}\right)^{\lfloor D/2 \rfloor - 1} \sum_{e \in E_x} P(\delta_e A_n), \tag{10}$$

$$sP(\delta_x A_n) \le \frac{2^{D-1}}{(1-s)^{D-2}} \sum_{e \in E_x} P(\delta_e A_n) \tag{11}$$

and use the fact that each bond in $E(n)$ obviously contains at most two end points in $\widetilde{V}(n)$.

To prove (10), first note that $sP(\delta_x A_n) = P(A_{x,n})$, where $A_{x,n}$ is the event that the site x is occupied and that $\delta_x A_n$ holds. We introduce a transformation F_x^B: $A_{x,n} \to \bigcup_{e \in E_x} \delta_e A_n$, with the following properties:

(i) $P(F_x^B)(\omega)) \ge (1-b)^{\lfloor D/2 \rfloor - 1} P(\omega)$.

(ii) Each element of $\bigcup_{e \in E_x} \delta_e A_n$ is the image of at most $2^{\lfloor D/2 \rfloor - 1}$ elements of $A_{x,n}$.

The existence of such a transformation clearly implies (10). To produce F_x^B, we suppose the bonds in E_x to be ordered in some arbitrary fashion and we proceed as follows.

Given $\omega \in A_{x,n}$, let $E_{x,\omega}^{\mathrm{in}}$ be the set of bonds in $E_x$that in the configuration ω, have the following property: Their end point distinct from x is either the root or else it is a site that is occupied and connected to the root by an internally occupied chain that does not pass through x. Let $E_{x,\omega}^{\mathrm{out}}$ be the set of bonds in E_x that, in the configuration ω, have the following property: Their end point distinct from x is either a site at distance $n+1$ from the root or else it is a site that is occupied and connected to a site at distance $n+1$ from the root by an internally occupied chain that does not pass through x. Whereas $\omega \in \delta_x A_n$, we have $E_{x,\omega}^{\mathrm{in}} \neq \emptyset$, $E_{x,\omega}^{\mathrm{out}} \neq \emptyset$ and $E_{x,\omega}^{\mathrm{in}} \cap E_{x,\omega}^{\mathrm{out}} = \emptyset$. If $|E_{x,\omega}^{\mathrm{in}}| \le |E_{x,\omega}^{\mathrm{out}}|$, set $E_{x,\omega} = E_{x,\omega}^{\mathrm{in}}$; otherwise, set $E_{x,\omega} = E_{x,\omega}^{\mathrm{out}}$. Whereas $|E_x| \le D$, obviously $|E_{x,\omega}| \le \lfloor D/2 \rfloor$. Define $F_x^B(\omega)$ as the configuration obtained from ω by vacating the first $|E_{x,\omega}| - 1$ bonds of $E_{x,\omega}$. In the configuration $F_x^B(\omega)$, the last bond in $E_{x,\omega}$ is pivotal for A_n, so that F_x^B indeed maps $A_{x,n}$ to $\bigcup_{e\in E_x}\delta_e A_n$. It is clear that also (i) holds. To see that (ii) holds, we can argue as follows. The transformation F_x^B can only modify the configuration at bonds incident to x. Therefore, by looking at $F_x^B(\omega)$ restricted to the other bonds and sites in $E(n)$ and $\widetilde{V}(n)$, we can find what the set $E_{x,\omega}$ is. If $F_x^B(\omega') = F_x^B(\omega'')$, ω' and ω'' can only differ at the bonds in $E_{x,\omega'} = E_{x,\omega''}$, with the exception of the last such bond. Because the cardinality of this set where they can differ is at most $\lfloor D/2\rfloor - 1$, this proves (ii).

To prove (11), we proceed in a similar fashion. We introduce a transformation F_x^S: $A_{x,n} \to \bigcup_{e\in E_x}\delta_e A_n$, with the following properties:

(i) $P(F_x^S(\omega)) \ge (1-s)^{D-2}P(\omega)$.

(ii) Each element of $\bigcup_{e\in E_x}\delta_e A_n$ is the image of at most 2^{D-1} elements of $A_{x,n}$.

The existence of such a transformation clearly implies (11). To produce F_x^S, we suppose the sites in V_x to be ordered in some arbitrary fashion, except for the fact that if the root is contained in V_x, it will be the last site in the order. Given $\omega \in A_{x,n}$, let $V_{x,\omega}^{\mathrm{in}}$ be the set of sites in V_x that in the configuration ω, have the following property. The site is either the root or else it is occupied, the bond that it shares with x is occupied and it is connected to the root by an internally occupied chain that does not pass through x. Whereas $\omega \in \delta_x A_n$, we have $V_{x,\omega}^{\mathrm{in}} \ne \varnothing$. Let $\overline{V}_{x,\omega}^{\mathrm{out}}$ be the set of sites in V_x that, in the configuration ω, are connected to a site at distance $n+1$ from the root by an internally occupied chain that does not pass through x. Set $\overline{V}_{x,\omega}^{\mathrm{in}} = V_x \setminus \overline{V}_{x,\omega}^{\mathrm{out}}$. Note that because $\omega \in A_{x,n}$, we have $\overline{V}_{x,\omega}^{\mathrm{out}} \ne \varnothing$, $V_{x,\omega}^{\mathrm{in}} \subset \overline{V}_{x,\omega}^{\mathrm{in}}$ and hence $|V_{x,\omega}^{\mathrm{in}}| \le |\overline{V}_{x,\omega}^{\mathrm{in}}| \le D-1$. Define $F_x^S(\omega)$ as the configuration obtained from ω by vacating the first $|V_{x,\omega}^{\mathrm{in}}|-1$ site of $V_{x,\omega}^{\mathrm{in}}$. Note that in the configuration $F_x^S(\omega)$, the bond in E_x, which has as x end points and the last site in $V_{x,\omega}^{\mathrm{in}}$, is pivotal for A_n, so that F_x^S indeed maps $A_{x,n}$ to $\bigcup_{e\in E_x}\delta_e A_n$. It is also again clear that (i) holds. To see that (ii) holds, we can argue as follows. The transformation F_x^S can modify only the configuration at sites in V_x. Therefore, by looking at $F_x^S(\omega)$ restricted to the other bonds and sites in $E(n)$ and $\widetilde{V}(n)$, we can identify the set $\overline{V}_{x,\omega}^{\mathrm{in}}$. If $F_x^S(\omega') = F_x^S(\omega'')$, ω' and ω'' can differ only at the sites in $\overline{V}_{x,\omega'}^{\mathrm{in}} = \overline{V}_{x,\omega''}^{\mathrm{in}}$. Whereas the cardinality of this set where they can differ is at most $D-1$, this proves (ii). $\square$

3. Proof of Theorem 1.4. Let

$$\mathscr{P}_G = \{(b,s)\in[0,1]^2 : \pi^{\mathrm{BS}}(b,s) = 1\} = \{(b,s)\in[0,1]^2 : \theta^{\mathrm{BS}}(b,s) > 0\}$$

be the percolative phase of the mixed percolation model on a graph G.

PROPOSITION 3.1. *Suppose that G is an infinite connected graph of degree bounded above by $D < \infty$. Suppose also that for some constants $C > 0$,*

$0 \le b_1 < b_2 \le 1$, $0 \le s_1 < s_2 \le 1$, *the inequality*

$$\frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial s} \le C \frac{\partial \tilde{\theta}_n^{\mathrm{BS}}(b,s)}{\partial b} \tag{12}$$

holds for arbitrary n *and arbitrary* $(b,s) \in [b_1,b_2] \times [s_1,s_2]$.

(i) *If* $(b,s) \in ([b_1,b_2] \times [s_1,s_2]) \cap \mathscr{P}_G$, *then for all* $x \ge 0$ *such that* $(b+x, s-x/C) \in [b_1,b_2] \times [s_1,s_2]$ *we have* $(b+x, s-x/C) \in \mathscr{P}_G$.

(ii) *If* $(b,s) \in ([b_1,b_2] \times [s_1,s_2]) \cap (\mathscr{P}_G)^c$, *then for all* $x \ge 0$ *such that* $(b-x, s+x/C) \in [b_1,b_2] \times [s_1,s_2]$ *we have* $(b-x, s+x/C) \in (\mathscr{P}_G)^c$.

PROOF. From (12), we have, under the conditions in (i),

$$\tilde{\theta}_n^{\mathrm{BS}}(b,s) \le \tilde{\theta}_n^{\mathrm{BS}}(b+x, s-x/C).$$

Therefore, claim (i) follows from (5).

The proof of (ii) is analogous. □

In combination with Proposition 2.1, Proposition 3.1 provides estimates on the behavior of the function $s_c(\cdot)$. To obtain estimates in the opposite direction, we can use the complementary inequalities (8) or (9). For our purposes, we can do better, though, using (1). This is done in the next proposition, in which we recall results and arguments from [15].

PROPOSITION 3.2. *Suppose that* G *is an infinite, locally finite, connected graph.*

(i) *If* $(b,s) \in \mathscr{P}_G$, *then for all* $0 < \alpha \le 1$ *such that* $(b\alpha, s/\alpha) \in [0,1]^2$ *we have* $(b\alpha, s/\alpha) \in \mathscr{P}_G$.

(ii) *If* $(b,s) \in (\mathscr{P}_G)^c$, *then for all* $0 < \alpha \le 1$ *such that* $(b/\alpha, s\alpha) \in [0,1]^2$ *we have* $(b/\alpha, s\alpha) \in (\mathscr{P}_G)^c$.

PROOF. To prove (i), consider the random graph G' obtained by performing mixed percolation on G with parameters $(b, s/\alpha)$. By performing site percolation on G' with parameter α (independently of the process that randomly originated G'), we obtain a random graph G'' that has the law of a random graph obtained from performing mixed percolation with parameters (b,s) on G. Because we suppose that $(b,s) \in \mathscr{P}_G$, we must have that G'' contains a.s. some infinite component. This means that G' supports site percolation at parameter α. From (1), if follows that G' also supports bond percolation at parameter α. In other words, the random graph G''' obtained by performing bond percolation on G' with parameter α (independently of the process that randomly originated G') contains a.s. some infinite component. However, G''' has the law of a random graph obtained from performing mixed percolation with parameters $(b\alpha, s/\alpha)$ on G. This proves (i).

The claim in (ii) follows from that in (i). □

PROOF OF THEOREM 1.4. This theorem is immediate from Propositions 2.1, 3.1 and 3.2. □

4. Proof of Theorem 1.1.

PROOF OF PART (a) OF THEOREM 1.1. We suppose, with no loss, that $b_c < 1$. We apply Proposition 3.1 with $b_1 = b_c$, $b_2 = (b_c + 1)/2$, $s_1 = 1/2$, $s_2 = 1$ and

$$C = \frac{2^{\lfloor D/2 \rfloor}}{s_1(1-b_2)^{\lfloor D/2 \rfloor - 1}} = \frac{2^{2\lfloor D/2 \rfloor}}{(1-b_c)^{\lfloor D/2 \rfloor - 1}}.$$

Note that this choice of C is possible thanks to Proposition 2.1.

Given $\varepsilon > 0$, set $b = b_c + \varepsilon$ and $s = 1$. If ε is small enough, then $(b, s) \in ([b_1, b_2] \times [s_1, s_2]) \cap \mathscr{P}_G$ and $x = (1/2)(1 - b_c) - \varepsilon > 0$. Whereas $x \le 1$ and $C \ge 2$, we also have $(b + x, s - x/C) \in [b_1, b_2] \times [s_1, s_2]$. Proposition 3.1(i) gives now $(b + x, s - x/C) \in \mathscr{P}_G$. This can be rewritten as

$$\left(\frac{1+b_c}{2}, 1 - \frac{(1-b_c)^{\lfloor D/2 \rfloor}}{2^{2\lfloor D/2 \rfloor + 1}} + \frac{\varepsilon}{C} \right) \in \mathscr{P}_G.$$

Because $\varepsilon > 0$ can be taken arbitrarily small, we obtain

$$s_c\left(\frac{1+b_c}{2} \right) \le 1 - \frac{(1-b_c)^{\lfloor D/2 \rfloor}}{2^{2\lfloor D/2 \rfloor + 1}}.$$

The proof is, therefore, complete, because $s_c = s_c(1) \le s_c((1 + b_c)/2)$. $\square$

PROOF OF PART (b) OF THEOREM 1.1. To construct our example, we first need to introduce various types of finite graphs to be used as our building blocks. Each one of these finite graphs has exactly two vertices, which are called its exterior vertices; the other vertices are called interior vertices.

A K bridge is a finite graph with $K + 2$ vertices, where two of the vertices are and the others are called interior vertices. The set of edges is defined by declaring that each one of the two exterior vertices is connected by edges to each one of the interior vertices. Note that the graph has $2K$ edges, that the two exterior vertices play the same role and have degree K, and that K interior vertices play the same role and have degree 2.

Suppose that $K_1 \le K_2$. A (K_1, K_2) double bridge is a graph obtained by identifying one of the exterior vertices of a K_1 bridge with one of the exterior vertices of a K_2 bridge. The interior vertices of the two original graphs, as well as the vertex resulting from the identification just described, are considered to be interior vertices of the new graph, whereas the other two vertices are considered to be exterior vertices of the new graph. Note that the vertex that results from the identification has degree $K_1 + K_2$, the other interior vertices have degree 2, one of the exterior vertices has degree K_1 (it will be called the first exterior vertex of the graph) and the other exterior vertex has degree K_2 (it will be called the second exterior vertex of the graph). (If $K_1 = K_2$, then the two exterior vertices play the same role and the choices of first and second are arbitrary.)

An l chain of (K_1, K_2) double bridges is a graph obtained as follows. Start with a sequence of l disjoint (K_1, K_2) double bridges. Now, for $i = 1, \dots, l - 1$, identify the second exterior vertex of the ith graph in the sequence with the first exterior vertex of the $(i+1)$th graph in the sequence. The interior vertices of the l original graphs, as well as the vertices resulting from the identifications just described, are considered to be interior vertices of the new graph, whereas the other two vertices (the first exterior vertex of the first graph and the second exterior vertex of the last graph) are considered to be exterior vertices of the new graph. The first exterior vertex of the first graph is called the first exterior vertex of the new graph, and the second exterior vertex of the last graph is called the second exterior vertex of the new graph. Note that the degrees of the exterior vertices are K_1 and K_2, whereas the degrees of the interior vertices are 2 and $K_1 + K_2$.

A decorated l chain of (K_1, K_2) double bridges is a graph obtained from an l chain of (K_1, K_2) double bridges by adding two new vertices, $v-$ and $v+$, the first of which is joined by an edge to the first exterior vertex of the l chain of (K_1, K_2) double bridges, and the second of which is joined by an edge to the second exterior vertex of the l chain of (K_1, K_2) double bridges. The vertices $v-$ and $v+$ are the only exterior vertices of the new graph, and are called, respectively, its first exterior vertex and second exterior vertex. Note that the degree of the exterior vertices is 1, whereas the degrees of the interior vertices are $K_1 + 1$, $K_2 + 1$, 2 and $K_1 + K_2$.

We are ready now to introduce an infinite graph that will serve as our example. Its maximal degree is D and for it to satisfy the desired relationship between s_c and b_c, a certain parameter l has to be taken large enough. For the moment, recall that $\lceil x \rceil$ is the smallest integer that is not smaller than x. Note that $\lfloor D/2 \rfloor + \lceil D/2 \rceil = D$. Let T_2 denote the binary tree, that is, the tree in which each vertex has degree 3. Our graph is denoted by $T_{2,D,l}$ and is obtained by replacing each edge of T_2 with a decorated l chain of $(\lfloor D/2 \rfloor, \lceil D/2 \rceil)$ double bridges, in the sense that the exterior vertices of the decorated l chains play the role of the vertices of T_2. To assure that the graph $T_{2,D,l}$ is quasitransitive, the orientation of the decorated l chains of $(\lfloor D/2 \rfloor, \lceil D/2 \rceil)$ double bridges is relevant, and a proper choice corresponds to identifying the second exterior vertex of each such decorated l chain with the first exterior vertices of two other such decorated l chains. Note that, indeed, the maximal degree of $T_{2,D,l}$ is D.

Before we proceed, recall that we know from branching theory that the critical value for bond percolation on T_2 is 1/2. This allows us to write for $T_{2,D,l}$,

$$\begin{aligned} &b_c^2\Bigl(1-(1-b_c^2)^{\lfloor D/2 \rfloor}\Bigr)^l\Bigl(1-(1-b_c^2)^{\lceil D/2 \rceil}\Bigr)^l \\ &\quad = \tfrac{1}{2} = s_c^{2+2l}\Bigl(1-(1-s_c)^{\lfloor D/2 \rfloor}\Bigr)^l\Bigl(1-(1-s_c)^{\lceil D/2 \rceil}\Bigr)^l. \end{aligned} \tag{13}$$

Note, in particular, that

$$\lim_{l\to\infty} b_c = 1, \tag{14}$$

because otherwise the 1.h.s. of (13) would become smaller than 1/2 for some large values of l.

Whereas $\lfloor D/2 \rfloor \le \lceil D/2 \rceil$ and $0 < s_c \le 1$, we can extract two inequalities from (13):

$$b_c^2 \Big(1 - (1 - b_c^2)^{\lfloor D/2 \rfloor}\Big)^{2l} \le \tfrac{1}{2} \le s_c^{2l}. \tag{15}$$

From the first inequality in (15) and from (14), we can conclude that for large l,

$$1 - (1 - b_c^2)^{\lfloor D/2 \rfloor} \le \left(\frac{1}{\sqrt{2} b_c}\right)^{1/l} \le (b_c)^{1/l}. \tag{16}$$

From (15) and (16), we obtain now

$$\begin{aligned} s_c &\ge (b_c)^{1/l}\Big(1 - (1 - b_c^2)^{\lfloor D/2 \rfloor}\Big) \ge \Big(1 - (1 - b_c^2)^{\lfloor D/2 \rfloor}\Big)^2 \\ &\ge 1 - 2(1 - b_c^2)^{\lfloor D/2 \rfloor} = 1 - 2(1 + b_c)^{\lfloor D/2 \rfloor}(1 - b_c)^{\lfloor D/2 \rfloor} \\ &\ge 1 - 2^{\lfloor D/2 \rfloor + 1}(1 - b_c)^{\lfloor D/2 \rfloor}. \end{aligned}$$ □

REMARK. We chose the binary tree in the preceding construction for convenience. This provided us with the particularly simple expression (13), on which the rest of the derivation was built. However, there is great flexibility in the choice of the underlying graph. For instance, if we use the hexagonal lattice instead, we produce a graph that is not only quasitransitive, but is also a periodic graph in the sense of [16]. The left-hand side of (13) is now equal to the critical point for bond percolation on the hexagonal lattice. The right-hand side of (13) can be bounded above and below by numbers that are in the interval (0,1) and that are, respectively, upper and lower bounds for the critical point of an arbitrary one-dependent bond percolation process on the hexagonal lattice (such bounds are standard Peierls estimates). This is sufficient to follow the steps in the foregoing proof, with minor modifications, and to derive also in this case the inequality between b_c and s_c claimed in Theorem 1.1(b), when l is large. (The hexagonal lattice has degree 3, which is important in case $D = 3$ to assure that the final graph has degree 3 as well. If $D \ge 4$, we can also use the Euclidean square lattice $\mathbb{Z}^2$ as reference graph in the construction.)

5. Proof of Theorem 1.3. In this section, the notation includes an extra subscript to indicate the graph to which it refers. In [1], the following result was proved via relationships between Potts and random-cluster models and comparisons of them with independent bond percolation models. For an arbitrary infinite, locally finite graph H,

$$b_{c,H} \le 1 - \exp(-J_{c,H}) \le \frac{q b_{c,H}}{1 + (q-1) b_{c,H}}. \tag{17}$$

Given an infinite connected graph G of bounded degree, let G_s be the random graph obtained from G by performing site percolation with parameter s on it. Then, a.s.,

$$J^S_{c,G}(s) = J^S_{c,G_s}, \qquad b_{c,G}(s) = b_{c,G_s}. \tag{18}$$

From (17) with $H = G_s$ and (18), we obtain

$$b_{c,G}(s) \leq 1 - \exp(-J^s_{c,G}(s)) \leq \frac{q b_{c,G}(s)}{1 + (q-1) b_{c,G}(s)}. \tag{19}$$

For the purpose of comparing this last display with (4), we rewrite (4) in terms of $b_{c,G}(s)$ [recall that $b_{c,G}(s)$ and $s_{c,G}(b)$ are inverse functions for $b_c \leq b \leq 1$, $s_c \leq s \leq 1$]:

$$1 - \frac{1}{C_1}(s - s_{c,G}) \leq b_{c,G}(s) \leq 1 - \frac{1}{C_2}(s - s_{c,G}). \tag{20}$$

From (19) and (20), we obtain the desired inequalities:

$$\frac{1}{qC_2}(s - s_{c,G}) \leq \exp(-J^S_{c,G}(s)) \leq \frac{1}{C_1}(s - s_{c,G}).$$

The values of the constants $A^S_1 = 1/(qC_2)$ and $A^S_2 = 1/C_1$ that appear in the statement of the theorem are obtained from bounds on C_1 and C_2 in (4) that derive from Propositions 2.1, 3.1(ii) and 3.2(i). □

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