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Vertex models and random labyrinths: phase diagrams for ice-type vertex models

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Abstract. We propose a simple geometric recipe which allows the deduction of phase diagrams for a general class of vertex models obeying the ice rule. The disordered phase maps onto intersecting loop models which are interesting in their own right and are related to other statistical mechanical models. This mapping is also useful for the understanding of some ordered phases of these vertex models as they correspond to the polymer loop models with cross-links in their vulcanized phase.

Keywords: integrable spin chains (vertex models), loop models and polymers, phase diagrams (theory)

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Contents

1. Introduction

Ice-type models were originally introduced in order to describe the properties of ice [1, 2], and they have been later generalized to represent other types of hydrogen-bonded crystals [3]. In ice, the oxygen atoms form a lattice with the coordination number of four. Each bond of the lattice contains a single hydrogen atom that is shifted from the middle point toward one of the neighbouring oxygen atoms. It has been conjectured, on the basis of local electro-neutrality, that precisely two hydrogen atoms are located near each oxygen atom with the other two being shifted away from it. This rule is known as the *ice rule*, and it can be graphically represented by placing polarization arrows along the hydrogen bonds. For each site on the lattice there are exactly two incoming and two outgoing arrows. There are six such possible arrow configurations at each site which leads to another common name for this model (or rather its two-dimensional square lattice version): the six-vertex model. Interestingly, the states of ice-type models can be characterized by topological winding numbers and the excitations are topological defects carrying fractional charge [4]; hence the recent revival of interest to this type of models in the context of topological order and fractionalization in quantum systems [5]–[8]. The zero-temperature quantum phase transition in the quantum square lattice six-vertex model has been argued to mimic the physics of the high-temperature superconducting materials with the d-density wave order while their counterparts on the triangular lattice twenty-vertex model could be relevant to the physics of 2D Wigner crystals [9].

The six-vertex model on the square lattice was solved exactly by Lieb [10]–[13] and Sutherland [14] using the Bethe ansatz and also (for some cases) by mapping it to a soluble free-fermion problem [15]. For a review see [16, 17]. Much less is known about the ice-rule models in other contexts. For example, in the 'next' 2D model, namely the twenty-vertex model on the triangular lattice, the Bethe ansatz only works in a few instances when there are special relations between the parameters [18]–[20]. More importantly, there are very few broad-based techniques of general applicability with which these sorts of models can be studied.

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Figure 1. The three possible switches for the two-dimensional square lattice. We do not distinguish between under- and over-crossings for the *gamma*-switches (we are not concerned with topology for the purposes of this paper).

In this paper we develop a new approach to establishing and understanding the phase diagrams of vertex models with the ice rule—meaning an arrow on each bond and equal numbers of incoming and outgoing arrows at each vertex. We introduce a new class of polymer models which are closely related to *loop models*, also a topic of intensive study (see [21] and references therein). While a lot of progress has been achieved in studying loops models by the means of exact solutions [22] and conformal field theory [23], in the present analysis we will not make recourse to those methods. The proposed general model is closely related to so-called *Lorentz lattice gas* [24]–[27] or *random labyrinths* [28] as well as the ice-rule vertex models. Indeed, as will be discussed below these are just special cases of a larger class of graphical models. The unified graphical representation on which our approach is based is not entirely new; certain aspects of it have been discovered in the context of a loop algorithm developed to simulate vertex models [29, 30]. Notwithstanding the computational benefits inherent in this representation, here we will concentrate on the physical insight into the problem and the theoretical benefits which it provides. Indeed, our perspective allows for a very intuitive approach to a general class of vertex models and opens new possibilities for their rigorous analysis. This approach is ideologically similar to that of Fortuin and Kasteleyn [31] for the case of the Potts model.

Since a lot is known about the square lattice six-vertex model, this is a good reference point from which to start. We begin by reviewing a connection between the six-vertex and an intersecting loop model and show how the phase diagram can be immediately inferred. We then apply our method to a particularly interesting case of the twenty-vertex model identifying its phase boundaries. A more complete treatment of this problem with necessary proofs and additional examples will be presented elsewhere [32].

2. The switch model

Let us first introduce a statistical model of objects that we call *switches*. Consider a finite lattice $\mathbb{L}$ with an even coordination number z. For simplicity let us discuss the case of a homogeneous lattice and, in particular, postpone the consideration of boundary sites. A *switch* is a variable associated with a lattice site. It is defined as a sorting of all incident edges into associated pairs. A more formal definition is as follows: consider the set $G(\mathbf{r})$ consisting of those bonds of $\mathbb{L}$ incident to the site $\mathbf{r}$. A switch $s(\mathbf{r})$ is a perfect matching (a paring) of the bonds in $G(\mathbf{r})$. Thus there are exactly $(z-1)!!$ switch types. On the square lattice, the three possible switches, the α-switch, the β-switch and the γ-switch, are shown in figure 1. In [28], in the context of random labyrinths, these have been referred to as NW and NE mirrors and tunnels respectively.

In general, we may call the switches the α_1-switch, the α_2-switch, etc. We will denote by $\mathcal{S}$ a configuration of such switches on $\mathbb{L}$; i.e. for each site $\mathbf{r} \in \mathbb{L}$, $s(\mathbf{r})$ is the assignment

of one of the $(z-1)!!$ such switch types s_k to this site. The weight of $\mathcal{S}$, which we denote by $Q(\mathcal{S})$ is given as follows: first, there are *a priori* weights (or activities) associated with every type of switch. These will be conveniently denoted by α_1, α_2, etc. Next, we observe that $\mathcal{S}$ divides the bonds of the lattice into loops. Indeed, each bond $\langle \mathbf{r}, \mathbf{r}+\mathbf{e} \rangle$ is a member of two switches $s(\mathbf{r})$ and $s(\mathbf{r}+\mathbf{e}) \in \mathcal{S}$ which pair it with two other bonds, etc. We can continue growing this path in both directions until the loop is closed, i.e. the two 'end' bonds are paired by a switch. A *loop* is thus a cycle of bonds such that every two adjacent bonds are paired up by a corresponding switch in $\mathcal{S}$. We are not distinguishing the cycles which differ only by a reversal of the overall order. Notice that in general such a loop can visit the same site more than once and in particular, due to tunnels, can have multiple self-intersections. We let $\ell(\mathcal{S})$ denote the number of such loops and $\mathrm{A}_1(\mathcal{S})$, $\mathrm{A}_2(\mathcal{S}), \ldots$ the number of α_1-switches, α_2-switches, etc. Then Q is given by

$$Q = \prod_{k=1}^{(z-1)!!} \alpha_k^{\mathrm{A}_k(\mathcal{S})} n^{\ell(\mathcal{S})} \tag{1}$$

where $(z-1)!!$ is the total number of distinct switches and n is a positive real number. Note, however, that if n is an integer, it can be thought of as the number of different colours available for every loop. The partition function is simply

$$Z = \sum_{\mathcal{S}} Q(\mathcal{S}). \tag{2}$$

We also introduce a simple extension of the above switch model in which we allow two (or more—depending on the lattice) paths at a given site to be 'fused' together. We shall refer to such objects as *cross-links*. In general, the cross-links may be *partial* in the sense that not all pairs of bonds adjacent to a site are fused. For a four-coordinated lattice such as the 2D square lattice, there naturally can be only a single type tying up all four adjacent bonds. Aside from associating the additional *a priori* weight ϕ with such cross-links, their introduction results in the following. If n is an integer, loops that have been fused together are now required to have the same colour. For an arbitrary n this translates into the requirement that all such fused loops count as one cluster for the purposes of evaluating the corresponding weights in equation (1).

As has already been mentioned, the switch model and its extension are closely related to several known statistical–mechanical models:

- The $n \to 0$ limit of the switch model describes the Eulerian walks or cycles (depending on the boundary conditions), i.e. walks that traverse every bond exactly once. Self-intersections are allowed via tunnels. If one were to adopt such a model for describing a polymer, the weight for the tunnels should translate into the rigidity of a polymer molecule.
- The $n = 1$ case corresponds to *Lorentz lattice gas* [24]–[27] or *random labyrinths* [28]. The model in [28] on a hypercubic lattice $\mathbb{Z}^d$ is nearly identical to our extended model, with the following difference: instead of cross-links, there are *normal sites* at which a path that is otherwise consistent with the switches can randomly, with equal probability, go in any of the $2d$ directions. This is equivalent to constructing a random walk on a graph whose vertices are the cross-links and edges are the switch-mediated paths between them. The model corresponds to $n = 1$ since the switches (mirrors

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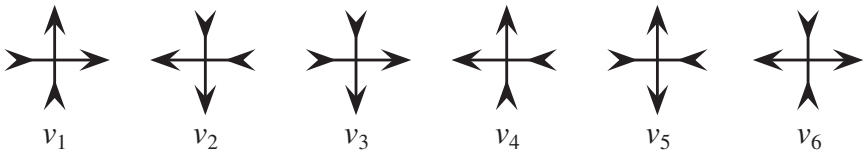

Figure 2. The six vertices allowed by the ice rule.

and tunnels) as well as cross-links (normal sites) are distributed with independent probabilities.

- For arbitrary n, if all switch weights are identical, this is just a fully-packed limit of the O(n) loop model [33].
- A special integrable case of a model (1) with $n < 2$ corresponds to a supersymmetric spin chain [34].
- Finally, the $n = 2$ case corresponds to the ice-rule vertex models; this is the subject of the following section.

3. Relation to the ice-type models

A mapping between $n = 2$ switch models and ice-type models is seen most easily in the case of the square lattice six-vertex model whose possible vertices v_i ($i = 1, \ldots, 6$) are shown in figure 2. Each of those vertices is assigned a corresponding energy ϵ_i. In the absence of external electric field, the model should be invariant under the reversal of all arrows, thus requiring $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$ and $\epsilon_5 = \epsilon_6$. For the remainder of this paper we shall concern ourselves not with the energies as such, but rather with the (unnormalized) Boltzmann weights $w_i \propto \exp(-\beta\epsilon_i)$. Conforming to the usual notations (cf [16, 17]), we shall use $a = w_1 = w_2$, $b = w_3 = w_4$ and $c = w_5 = w_6$. The weight for each legitimate vertex configuration is given by the product of the weights of all participating vertices.

The mapping of the switch model to the six-vertex model is done by assigning directions to all loops (since there are two choices for every loop, the factor of 2^ℓ in equation (1) can be obtained by summing over all such choices). Tracing out the switch variables as illustrated in figure 3(a) leaves us with the six-vertex configurations[4]. The weights are given by

$$a = \beta + \gamma, \qquad b = \alpha + \gamma, \qquad c = \alpha + \beta. \tag{3}$$

In general, we claim that the partition function for the switch model with weights (1) and $n = 2$ is exactly the partition function for an ice-rule vertex model with vertex weights determined by

$$w_i = \sum_k \alpha_k \Delta_i^k \tag{4}$$

where $\Delta_i^k = 1$ if the vertex v_i and the switch s_k are *consistent* and is 0 otherwise. Here v_i and s_k are *consistent* if there is one incoming and one outgoing arrow for each pairing

[4] An analogous construction for the case of square ice ($a = b = c$) was described in our earlier paper [33]. In fact, similar ideas have been discussed in [35]–[37], [29, 38, 39, 30], usually confined to special cases. However, with the exception of [37, 38]), they have not been used as a basis for an *analytical* approach.

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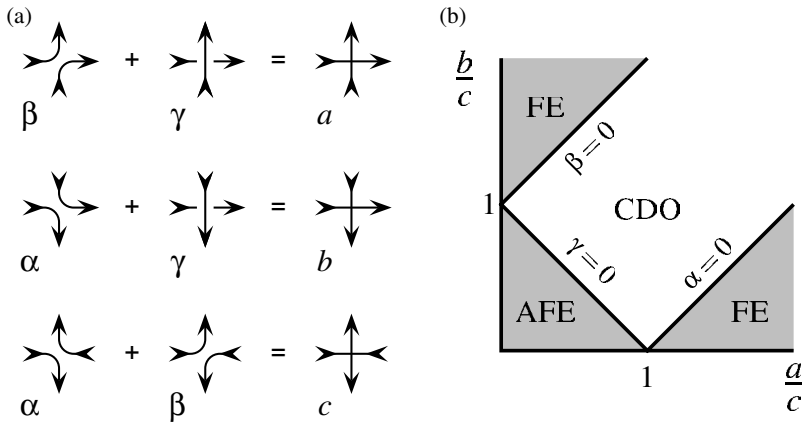

Figure 3. (a) Tracing out switches by fusing directed loop segments into the ice-type vertices (only types 1, 3 and 5 are shown; types 2, 4 and 6 are obtained by merely reversing all arrows). (b) The phase diagram of the six-vertex model.

of the switch. (Notice that this automatically preserves the ice rule.) The most direct way to see this is to consider a joint measure on both switch configurations $\mathcal{S}$ and arrow configurations $\mathcal{A}$ with the weights

$$\tilde{Q} = \prod_{k=1}^{(z-1)!!} \alpha_k^{\mathrm{A}_k(\mathcal{S})} \Delta(\mathcal{S}, \mathcal{A}) \tag{5}$$

where $\Delta(\mathcal{S}, \mathcal{A}) = 1$ if $\mathcal{S}$ and $\mathcal{A}$ are consistent in the above sense and 0 otherwise. Now, summing this weight over all possible arrow configurations $\mathcal{A}$ leads to the weight of $\mathcal{S}$ given by equation (1) while tracing out switches leaves us with ice-type arrow configurations where the weight for a single vertex is given by equation (4).

We now take a closer look at equations (3) and notice that if $\alpha, \beta, \gamma > 0$ (and the switch model, we emphasize, only makes sense for non-negative weights), the allowed values of a, b and c lie precisely within the entire critical disordered (CDO) region of the six-vertex model (figure 3(b)). In fact, we argue, this is not a coincidence but a general case: a switch model with *all* switch weights positive always maps onto a disordered phase of a vertex model[5].

What about the order parameter for the switch model? For integer n, it seems that *ordering* can be associated with the breaking of colour symmetry. Thus, a natural choice would be the excess probability, above the *a priori* $1/n$, that the bond at the origin is of a given favoured colour. Without detailed consideration of the behaviour on the boundary, e.g. on a torus, such an order parameter may vanish for spurious reasons. This can be cured by the insertion of a small external colour field or by specific rules at the boundary of $\mathbb{L}$. Here, in particular, we may invoke the rule that all paths terminating at the boundary are identified as part of single loop (or cluster) which happens to be of the favoured colour. In these circumstances, the order parameter is exactly the probability that the bond at the origin is connected to the boundary—analogous to the situation in

[5] This is never a truly disordered phase; due to conserved 'arrow currents', the correlations are always expected to be critical (even in higher dimensions) as is the case for the six-vertex model.

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the Fortuin–Kasteleyn (FK) random cluster representation of the q-state Potts model. For non-integer n, this can be taken over as the *definition* of the order parameter.

While there is no general relation between the percolation (colour-based) order parameter and the physical (arrow-based) order parameters, in specific instances they *can* be related. (Compare this with the FK random cluster models where the probability of connection to the (wired) boundary is simply *equal* to the spontaneous magnetization.) For example, if $\gamma = 0$, then the staggered polarization order parameter associated with the AFE phase is the aforementioned colour order parameter, while the probability that two bonds are in the same loop (or cluster, if the cross-links are present) represents the relevant correlation function. Similar statements hold for the uniform polarization if $\alpha = 0$ or $\beta = 0$. Unfortunately, all of the above equivalences break down when α, β and γ are all positive; these formulae must be modified by the switch content of the relevant paths[6]. Notwithstanding, the perspective of this note is that colour symmetry breaking is the key to understanding these systems. For example, in the six/eight-vertex model context, there is an interesting conventional-type transition with standard types of critical indices and relations there among as the cross-link weight ϕ increases from zero.

Let us now present an appealing (albeit non-rigorous) argument that any $n \geq 1$ switch model with all switch weights positive is always in a colour-symmetric phase. According to our previous discussion, here all we need do is establish that only a vanishing fraction of the bonds is connected to the boundary. Our principal argument will be for $n = 1$ where the switches may be placed independently. Consider a half-space problem where, using their independence, the system may be grown upward layer by layer starting from the boundary. In such a setting, it is clear that the assumption of a positive density of vertical bonds connected back to the boundary is simply not plausible because any nearby pair can be terminated with uniform probability. For example, in $d = 2$, if two strands are separated by a few horizontal edges, turning the left strand to the right with an α-switch and the right one to the left with a β-switch along with the convenient placement of γ-switches on the intervening horizontal bonds terminates both strands.

For the $n > 1$ cases, we now argue that in general longer strands are more heavily suppressed, which only strengthens the previous conclusion. Of course, these considerations do not preclude a power-law or even logarithmic decay of such density away from the boundary, and in fact this is precisely what has been observed numerically [25] for $n = 1$; for $n \gg 1$ it has been rigorously shown that the long loops are exponentially suppressed [33][7]. Finally notice that the above arguments are not restricted to 2D; we expect our criterion to work in any dimensionality.

Let us now turn to the ordered phases. As follows from the above discussion, a long-range order cannot appear unless one of the switch weights vanishes. To see that this condition indeed correctly describes the boundary, it is instructive to look at the six-vertex model again. Let us start with the FE phase boundary, e.g. the one described

[6] However, in all cases studied, the colour correlation function is an upper bound on the conventional one.

[7] We should mention here that at least in 2D, such loop model might still order if n becomes sufficiently large: in our earlier paper [33] we actually proved this for a wide class of models that include the switch model on the square lattice with $\alpha = \beta$. This ordering, however, happens not because the loops become long but rather because they become extremely short. When the entropy associated with large n forces most of the loops to become of length 4, they break the translational symmetry of the lattice by circling plaquettes in a checkerboard pattern. Naturally, such transition is not associated with any colour symmetry breaking.

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by $b = a + c$, which in our language is simply $\beta = 0$. Notice that a line going up can now either continue going up passing through γ-switches or turn to the left via an α-switch, whereupon it will continue to the left until the next α-switch sends it up again. But it can never turn back on itself. Thus, the total number of loops ℓ in equation (1) in an $L \times L$ sample is at most $2L$ and scales sublinearly with the size of the system $N = L^2$. As a result, in the thermodynamic limit, the factor of 2^ℓ in equation (1) is inconsequential and can be dropped. Then the partition function is trivially calculated as

$$Z = \sum_{\mathcal{S}\in\{\alpha,\gamma\}^N} \alpha^A \gamma^{N-A} = (\alpha + \gamma)^N = b^N \tag{6}$$

indicating that the system has become fully frozen: either all vertices are of type 3 or of type 4. Notice that, once again, this argument is not specifically 2D and could be easily used in higher dimensions; for example, it generalizes the proof by Nagle [40] of a first-order transition in a KDP model on d-dimensional tetrahedral lattices.

Inside the frozen phase, i.e. when $b > a + c$, the loop model can be extended by the addition of cross-links. The result is simply $b = \alpha + \gamma + \phi$, $a = \gamma$, $c = \alpha$ (this is in fact similar to the 'freezing' of a vertex in the loop algorithm of [29]).

However, it should be noted that in general a model with cross-links is not automatically equivalent to an ice-type vertex model. In particular, one has to verify that a consistent arrow assignment can be made for all of the cross-linked clusters with the cross-link vertices corresponding to the 'excessive' vertex type. For the FE phase, the assignment is automatic but not particularly enlightening since the model is already frozen at the phase boundary. The AFE phase presents a more interesting case. At the boundary of the AFE phase $\gamma = 0$, i.e. loops turn at every step and cannot self-intersect. Assigning directions renders each such loop consistent with a perfect staggered AFE order, but since each loop is free to choose one of the two possible directions, no overall order results. Moving into the phase, we can once again resort to cross-links to compensate for excessive c: $c = \alpha + \beta + \phi$, $a = \beta$, $b = \alpha$. Then, it turns out, the cross-links do not change the geometry of loops but rather 'vulcanize' them into clusters. Indeed, as is easy to verify, such clusters are always consistent with one of the two possible staggered polarizations. For this model, it is essential that in the CDO phase the loops themselves are critical (as expected in this type of models on general grounds). Therefore, at the AFE phase boundary, such vulcanization leads to percolation of a particular colour or, alternatively, staggered polarization throughout the sample. Similar—but trivial—considerations apply at the FE boundary. However, for n large, the considerations of [33] demonstrate that the analogue of the AFE transition does *not* occur at $\phi = 0$; the colour percolation transition occurs at some strictly positive ϕ where, presumably, it is first order.

4. Twenty-vertex model on the triangular lattice

So far we dealt with the lattice of coordination number four. This is somewhat special since the number of possible vertices (up to the reversal of all arrows) allowed by the ice rule is the same as the number of possible switches—three. This is not generally the case. Let us turn our attention to the triangular lattice. Its coordination number is six which means there are $\binom{6}{3} = 20$ distinct ways of arranging three inward and three outward pointing arrows. By requiring that the weights for the vertices are arrow-reversal

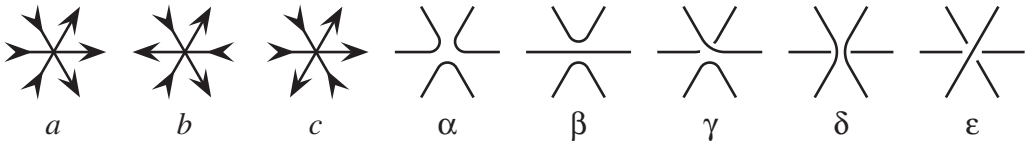

Figure 4. The distinct (up to rotations and reflections) vertices and switches on the triangular lattice with their weights. The third vertex is chiral, with only one chirality shown; both have the same weight.

invariant (zero electric field condition), the number of parameters is reduced to ten. On the other hand, there are $5!! = 15$ distinct switches. Therefore equations (4) form an under-determined system. So, at a first glance the situation appears hopeless: we cannot even uniquely solve this system, so how could we expect to deduce the phase boundaries? Nevertheless, the proposed criterion stands: as long as it is *possible* to satisfy equations (4) with positive switch weights, the system is in its (presumably critical) disordered phase. To illustrate this, consider the symmetric version of the twenty-vertex model (dubbed the F-model in [18]) by requiring that all vertex weights are invariant under the symmetry group D_6. This leaves us with only three distinct types of vertices and five types of switches as depicted in figure 4. In this case, equations (4) read

$$\begin{aligned} a &= \beta + 2\gamma + 2\delta + \epsilon, \\ b &= 2\alpha + 3\beta + \epsilon, \\ c &= \alpha + \beta + 3\gamma + \delta, \end{aligned} \tag{7}$$

or, equivalently

$$\begin{aligned} a + 2c &= b + 8\gamma + 4\delta, \\ b + 2c &= a + 4(\alpha + \beta + \gamma), \\ b + 3a &= 2c + 4(\beta + \delta + \epsilon). \end{aligned} \tag{8}$$

From the latter form, it is clear that there are no solutions with all switch weights positive unless $a + 2c > b$, $b + 2c > a$, $b + 3a > 2c$; according to our criterion this must be the CDO region of the model, see figure 5. The suspected phase boundaries correspond to $\alpha = \beta = \gamma = 0$, $\gamma = \delta = 0$ and $\beta = \delta = \epsilon = 0$. These conditions have a simple geometric interpretation in terms of loops: in the first case, connections of bonds at 60° are excluded, in the second, the are no connections at 120°, and in the third, there are no direct tunnels. We now turn to the suspected ordered phases. The simplest one is the phase II ($b > a + 2c$); it is possible to verify that the introduction of complete cross-links (ϕ_* in figure 6(a)) corresponding to excessive b-vertices is consistent with the onset of the spontaneous sextipole electric moment—a triangular flux phase. This is the only ordered phase of the model accessed by the means of an exact solution [18, 16] which is available along the dashed line in figure 5; a transition to the ordered phase was found to be of the infinite order.

In region I ($a > b+2c$) it can be shown that complete cross-links (ϕ_* in figure 6(a)) are not consistent with the dipolar order described by the proliferation of a-vertices. While the introduction of partial cross-links ($\phi_\times$ in figure 6(a)) alleviates some of the problem, it does not eradicate it altogether. Thus, a complete description of the vertex model in

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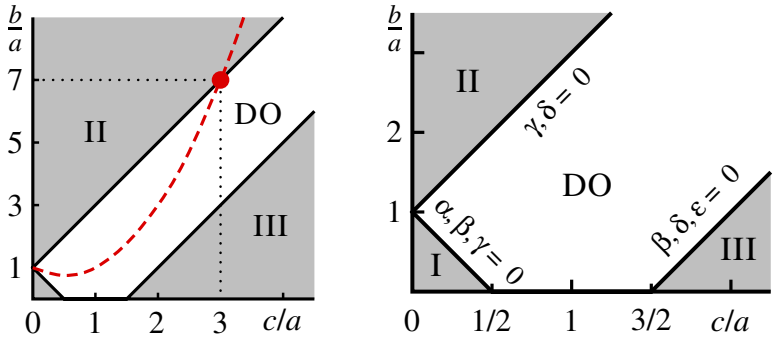

Figure 5. The proposed phase diagram for the F-model on the triangular lattice shown. Along the dashed line the Bethe ansatz solution is available with the known transition point circled. The blown-up neighbourhood of the origin, including region I, is shown on the right.

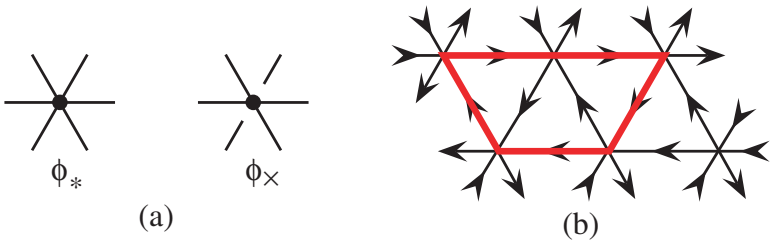

Figure 6. (a) Complete and partial cross-links relevant for understanding the ordered phases II and I (see text for details). Only one of the three possible partial cross-links is shown; the others can be obtained by its ±60° rotations. (b) A zero mode in a state composed entirely of *c*-vertices; the arrows along the selected pentagon can be freely reversed.

region I in terms of switches and cross-links remains open. Similar—and even worse—problems plague the model in region III ($2c > b + 3a$) where there is an abundance of *c*-vertices. Here, one must account for the presence of *local* zero modes (see figure 6(b)). Thus, in its unamended form, the proposed method certainly cannot describe all the rich physics of this complicated system; perhaps an additional idea is needed or, perhaps, the current method only works when the order parameter represents an excess of one of *two* possible ground state patterns.

However, we find the above picture of the phase boundaries and their relation to switch weights appealing. By analogy to the corresponding phase diagram on the square lattice—where the results are irrefutable—it is extremely likely that figure 5 indeed correctly identifies the phases of the F-model on the triangular lattice.

5. Twenty-vertex model on the cubic lattice

Interestingly, there is a one-to-one correspondence between the vertices (and corresponding switches) in twenty-vertex models on the 2D triangular and 3D cubic lattice (it becomes obvious if one identifies ±60° directions in figure 4 with *y*- and *z*-axes). Reducing the

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cubic lattice model to its F-model version by requiring that all vertex weights are invariant under O_h (the point symmetry group of the cubic lattice), we are left with only two vertex weights a and c (in notations of figure 4, $b \equiv a$) and three switch weights α, β and ϵ ($\gamma \equiv \alpha$ and $\delta \equiv \beta$). It then follows from equations (4) that $2a = c + 4\beta + 2\epsilon$. If $2a > c$, the model is in the DO phase. At $2a = c$ we have $\beta = \epsilon = 0$, meaning that loops must turn at every step; this is a 3D analogue of non-self-intersecting loops. Every loop is perfectly AFE-ordered, and as we have seen, on a 2D square lattice this signifies a transition to the AFE phase. There is also a general statement about self-intersections being a relevant perturbation in a similar type of 2D loop models [41] suggesting that the non-intersecting fully-packed loops characterize an unstable fixed point. So once again, it is suggestive that $2a = c$ is a transition point. In 3D, however, the region with $2a < c$ is analogous to region III of a triangular lattice model. In particular, there are zero modes in some of its ground states similar to that shown in figure 5 (right) (naturally, these modes corresponds to the loops of even length now). Thus, sorting out the properties of region III of the F-model on the triangular lattice might also shed some light on the properties of the cubic F-model.

6. Conclusion

To conclude, we have introduced a class of intersecting polymer loop models with the loop fugacity n. For $n = 2$, such a model is equivalent to an ice-type vertex model in its disordered critical phase. We conjecture that where the loops are critical, the boundaries of the disordered phase are encountered at the parameter values which force one or more switch weights to vanish. The onset of a long-range order in a vertex model is characterized by the coalescence of loops into a percolating cluster—an analogue of a vulcanization transition in polymers. This can often (but not always) be implemented by the introduction of cross-links into the switch model. This correspondence is *not* strictly 2D, although in this paper we have concentrated mostly on 2D examples to benefit from the additional information available from the exact solutions. We note that all known transition points in the ice-rule models satisfy our criterion. As an additional benefit, we can understand transitions on irregular lattices[8].

However, despite its suggestive nature, it remains a big challenge to turn our criterion into a rigorous statement.

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