Conformal Invariance for Certain Models of the Bond–Triangular Type

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Abstract: Convergence to SLE_6 of the percolation exploration process for a correlated bond–triangular type model studied in [5] is established, which puts the said model in the same universality class as the standard site percolation model on the triangular lattice [12]. The result is proven for all domains with boundary (upper) Minkowski dimension less than 2, following the general streamlined approach outlined in [11].

Keywords: Universality, percolation, conformal invariance.

1 Introduction

In this note, we investigate the percolation type model invented in [5] and prove the existence of the scaling limit for the associated exploration process (the model will be briefly reviewed in subsection 2.1). As anticipated, the limiting process is described by chordal SLE_6 – which coincides with the well–known limiting behavior of ordinary site percolation on the triangular lattice ([12], [4]). The result, while not unexpected, establishes for the first time a non–trivial example of universality of the sort anticipated since the 1960s: in particular, via the common continuum limit, the behavior of the present model on the lattice at long distances is asymptotically identical to that of the critical triangular site percolation model.

In the previous work [5], Cardy's formula was established for the model under consideration – at least for a certain class of domains. According to the methodology outlined by S. Smirnov in [11], to be recapitulated in Section 2.5, the existence of this conformal invariant in a model of the percolation–type should, in principle, lead directly to the above–mentioned scaling limit. While by and large this is the substance of the present work, many of the technical results necessary to implement this program were not *a priori* available. Specifically, tightness estimates which ensure the existence of limiting paths require an elaborate proof and, further, Cardy's formula for the system at hand has to be established in greater generality than had been done in [5]. Finally, needless to say, it is necessary to actually define an exploration process at the microscopic level. All of this will be accomplished via a sequence of constructions and lemmas, as outlined below, some of which may be of independent interest.

The paper is organized as follows: In §2.1 we will provide a concise description of the model, along with a definition of the associated exploration process. In §2.2 we provide background discussion – mostly to establish notation – for the continuum SLE_κ, which represents the ultimate objective for the relevant discrete process which is defined in §2.3. In §2.4 we will state the main theorem and the following technical results: (1) tightness, (2) preservation of admissibility, and (3) the requisite extension of Cardy's Formula from [5]. In §2.5 we will describe how these results are assembled into a proof of the main theorem. Finally in Section 3, we provide proofs for the above–mentioned technicalities.

2 Setup and Definitions

2.1 Review of Model

Here we give a quick description of the model under study. For more details see Section 2.2 of [5]. The model takes place on the hexagon tiling of the 2D triangular site lattice: hexagons are yellow, blue and sometimes split; half and half. Our description of the model starts with a particular local arrangement of hexagons:

Definition 2.1. A *flower* is the union of a particular hexagon with its six neighbors. The central hexagon we call an *iris* and the outer hexagons we call *petals*. We number the petals from 1 to 6, starting from the one directly to the right of the iris.

Let $\Omega \subset \mathbb{C}$ be a domain, which for simplicity we may regard as being a finite connected subset of the hexagon lattice, with certain hexagons being designated as irises (this determines the flowers) which is known as a *floral arrangement*. There are three restrictions on placement of irises: (i) no iris is a boundary hexagon, (ii) there are at least two non–iris hexagons between each pair of irises, and (iii) ultimately in infinite volume the irises have a periodic structure with 60° symmetries.

We are now ready to define the statistical properties of our model.

Definition 2.2. Let Ω be a domain with floral arrangement $\Omega_{\mathfrak{F}}$.

- Petals and hexagons in the complement of flowers are only allowed to be blue or yellow with probability 1/2.
- Irises can be blue, yellow, or mixed (one of three ways c.f. Figure 1) with probabilities a, a, or s, so that $2a + 3s = 1$ and in addition,

$$a^2 \geq 2s^2.$$

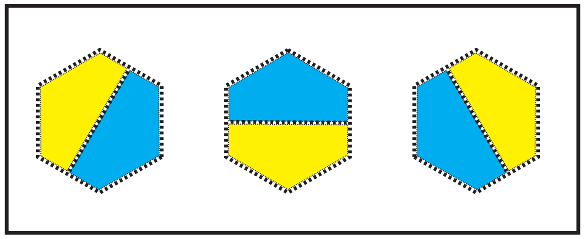

Figure 1: The three allowed "split" states of the hexagon. Note that these correspond to single bond occupancy events in the corresponding up–pointing triangle in the bond–triangular lattice percolation problem.

- The exceptional configurations on the flowers, which we call *triggers*, are configurations where there are three yellow and three blue petals, with one pair of blue (and hence also yellow) petals contiguous. In these configurations, the irises can now only be blue or yellow, each with probability 1/2.

Note that triggering is the only source of (very short range) correlation in this model. Everything else is configured independently.

Finally, it is remarked that the total of five possible configurations on a hexagon correspond to the eight possible configurations on (up–pointing) triangles – of which there are five distinct connectivity classes. It is not hard to see, by checking local connectivity properties, that the model described is a representation of a correlated percolation model on the triangular bond lattice.

It was shown in [5] Theorem 3.10 that our model exhibits all the typical properties of a 2D percolation model at criticality. Cardy's formula for this model was the main result of [5] (Theorem 2.4). More specifically, let $\Omega \subset \mathbb{C}$ be a domain with piecewise smooth boundary which is conformally equivalent to a triangle. Let us denote the three boundaries and prime ends of interest by $\mathcal{A}, e_{AB}, \mathcal{B}, e_{BC}, \mathcal{C}, e_{AC}$, in counterclockwise order. We endow Ω with an approximate discretization (with hexagons) on a lattice of scale $\varepsilon = 1/N$ and a floral arrangement $\Omega_{\mathfrak{F}_\varepsilon}$. Let z be the vertex of a hexagon in $\Omega_{\mathfrak{F}_\varepsilon}$. We define the discrete crossing probability function $U_\varepsilon^Y(z)$ to be the indicator function of the event that there is a blue path connecting $\mathcal{A}$ and $\mathcal{B}$, separating z from $\mathcal{C}$, with similar definitions for $V_\varepsilon^Y(z)$ and $W_\varepsilon^Y(z)$ and the blue versions of these functions. Then taking the scaling limit in an appropriate fashion (for more details see Section 2.3 of [5]), we have, e.g.

$$\lim_{\varepsilon\to 0} U_\varepsilon^Y = h_{\mathcal{C}},$$

where $h_{\mathcal{C}}$ is a so–called Carleson–Cardy function: It is harmonic, and on the up–pointing equilateral triangle with base $\mathcal{C}$ being the unit interval, it is equal to $\frac{2}{\sqrt{3}} \cdot y$ which satisfies Cardy's formula. The functions $h_{\mathcal{A}}$ and $h_{\mathcal{B}}$ are defined similarly.

2.2 SLE: Definitions and Notations

As the title of this subsection indicates, we will briefly review the relevant notions of Löwner evolution – mostly for the purpose of fixing notation. Let Ω be a domain with two boundary prime ends a and b.

Definition 2.3. Let $\{\Omega_t\}_{t=0}^{\infty}$ be a strictly decreasing family of subdomains of Ω ($t \in [0,\infty)$) which is Carathéodory continuous with respect to b, such that $\Omega_0 = \Omega$ and $\cap_{t=0}^{\infty}\Omega_t = b$. Then we call $\{\Omega_t\}_{t=0}^{\infty}$ a Löwner chain.

Let $\mathbb{H}$ denote the upper–half plane of $\mathbb{C}$. We can select some conformal map $g_0 : \Omega \to \mathbb{H}$ such that $g_0(a) = 0$ and $g_0(b) = \infty$. The family of conformal maps $g_t : \Omega_t \to \mathbb{H}$ normalized such that $g_t(b) = \infty$ and $g_0 \circ g_t^{-1}(z) = z + \frac{A(t)}{z} + o(1/z)$ are continuous in t. We now reparameterize time so that $A(t) = 2t$.

We call γ a *crosscut* in Ω from a to b if it is the preimage of a non–self–crossing curve from 0 to ∞ in $\mathbb{H}$ under g_0. Note that γ is allowed to touch itself but not to cross itself. We define Ω_t to be the connected component of $\Omega \setminus \gamma_{[0,t]}$ containing b. It's easy to see that Ω_t is a Löwner chain if and only if the following two conditions are satisfied for every $t > 0$:
(L1) $\gamma_t \in \overline{\Omega_{t-\varepsilon}}, \quad \forall \varepsilon > 0$
and
(L2) $\exists \delta_n \to 0, \quad \forall \varepsilon > 0, \quad \gamma_{t-\delta_n} \in \Omega_{t-\delta_n-\varepsilon}$.
Under these conditions, we can reparametrize γ so that the maps g_t's satisfy the following celebrated Löwner equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t},$$

where $\lambda_t = g_t(\gamma(t))$ is a continuous function.

On the other hand, the solution of the Löwner equation for any initial conformal map $g_0 : \Omega \to \mathbb{H}$ and any continuous function $\lambda(t)$ defines a Löwner chain, but not necessarily a curve (see [8] for a complete discussion). λ_t is called the *driving function* of Ω_t.

If we take the very special function $\lambda_t = B(\kappa t)$, where $B(t)$ is one-dimension Brownian motion started at zero, then the corresponding random Löwner chain is called the Stochastic (or Schramm) Löwner Evolution with parameter κ, SLE_κ. It is known that at least for Jordan domains it is a.s. generated by a random curve (see [10]). We would be particularly interested in the case $\kappa = 6$.

2.3 The Exploration Process

We now give a (microscopic) definition of the percolation exploration process tailored to our system at hand. We must start with a precise prescription of how to construct our domains.

We start with a bounded domain $\Omega \subset \mathbb{C}$ which has $M(\partial\Omega) < 2$. In the preceding, $M(S)$ denotes the (upper) Minkowski dimension of the set S which, as usual, is defined as

$$M(S) = \limsup_{\vartheta \to 0} \frac{\log \mathcal{N}(\vartheta)}{\log(1/\vartheta)},$$

where $\mathcal{N}(\vartheta)$ is the number of boxes of side length ϑ needed to cover the set. Let a and b be two prime ends and consider hexagons of the ε–tiling of $\mathbb{C}$. It is assumed that within this tiling (with fixed origin of coordinates) the locations of all irises/flowers/fillers are predetermined. We define Ω_ε to be the union of all fillers and flowers whose closure lies in the interior of Ω. It is assumed that ε is small enough that both a and b are in the same lattice connected component of the tiling. Other components, if any, will not be discarded but will only play a peripheral rôle. With the exception of flowers, the boundary of the domain will be taken as the usual internal lattice boundary, which consists of the points of the set which have neighbors not belonging to the set. If the lattice boundary cuts through a flower, then the whole flower is included as part of the boundary. The notation for this lattice boundary will be $\partial_\varepsilon\Omega_\varepsilon$.

Consider points $a_\varepsilon, b_\varepsilon$ which are on $\partial_\varepsilon\Omega_\varepsilon$ and are vertices of hexagons. We call $(\Omega_\varepsilon, \partial\Omega_\varepsilon, a_\varepsilon, b_\varepsilon)$ *admissible* if

- Ω_ε contains no partial flowers.
- $\partial_\varepsilon\Omega_\varepsilon$ can be decomposed into two lattice connected sets consisting of hexagons and/or semi–irises, one of which is colored blue and one of which is colored yellow, such that a_ε and b_ε lie at the points where the two sets join and such that the blue and yellow paths are legal paths in the sense of our model; in particular, the coloring of these paths do not lead to flower configurations that have probability zero.
- a_ε and b_ε lie at the vertices of hexagons, such that of the three hexagons sharing the vertex, one of them is blue, one of them is yellow, and the third is in the interior of the domain. (See Figure 2).

We remark that in the case of boundary flowers (and other sorts of clusters on the boundary) it is not necessary to color *all* the hexagons/irises. Indeed the coloring scheme need not be unique – it is only required that a boundary coloring of the requisite type can be selected.

It is not hard to see that the domains $(\Omega_\varepsilon, \partial_\varepsilon\Omega_\varepsilon, a_\varepsilon, b_\varepsilon)$ converges to $(\Omega, \partial\Omega, a, b)$ in the sense that $\partial_\varepsilon\Omega_\varepsilon$ and Ω_ε converge respectively to $\partial\Omega$ and Ω in the Hausdorff metric, and there exists a_ε and b_ε which converge respectively to a and b as $\varepsilon \to 0$. Notice that the latter convergence is really in terms of the preimages under the Riemann map of the relevant domain.

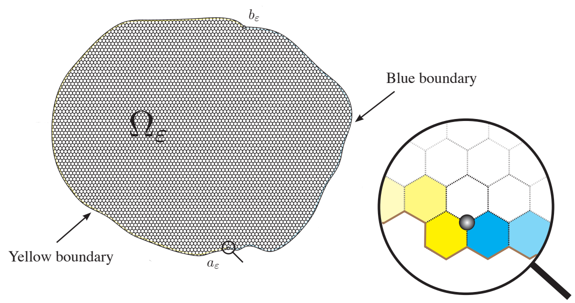

Figure 2: The setup for the definition of the exploration process.

Geometrically, the *exploration process* produces, in any percolation configuration on Ω_ε, the unique interface connecting a_ε to b_ε, i.e. the curve separating the blue lattice connected cluster of the boundary from that of the yellow. We denote this interface by γ^ε. Dynamically, the exploration *process* is defined as follows: Let $\mathbb{X}_0^\varepsilon = a_\varepsilon$. Given $\mathbb{X}_{t-1}^\varepsilon$, it may be necessary to color new hexagons in order to determine the next step of the process. (In particular, $\mathbb{X}_{t-1}^\varepsilon$ is "usually" at the vertex of a hexagon which has not yet been colored.) We color any necessary undetermined hexagons according to the following rules:

- If the undetermined hexagon is a filler hexagon, we color it blue or yellow with probability 1/2.
- If the undetermined hexagon is a petal or an iris, we color it blue or yellow or mixed with the conditional distribution given by the hexagons of the flower which are already determined.
- If a further (petal) hexagon is needed, it is colored according to the conditional distribution given by the iris and the other hexagons of the flower which have already been determined.

We are now ready to describe how to determine $\mathbb{X}_t^\varepsilon$:

- If $\mathbb{X}_{t-1}^\varepsilon$ is not adjacent to an iris, $\mathbb{X}_t^\varepsilon$ will be equal to the next hexagon vertex we can get to in such a way that blue is always on the right of the segment $[\mathbb{X}_{t-1}^\varepsilon, \mathbb{X}_t^\varepsilon]$.
- If $\mathbb{X}_{t-1}^\varepsilon$ is adjacent to an iris, then the state of the iris is determined as described above, after which the exploration path can be continued (keeping blue on the right) until a petal is hit. The color of the petal will now be determined (according to the proper conditional distribution) and $\mathbb{X}_t^\varepsilon$ will equal one of the two possible vertices common to the iris and the new petal which keeps the blue region to the right of the final portion of the segments joining $\mathbb{X}_{t-1}^\varepsilon$ to $\mathbb{X}_t^\varepsilon$

In particular, it is noted that at the end of each step, we always wind up on the vertex of a hexagon. We denote by γ_t^ε the actual value taken by the random variable $\mathbb{X}_t^\varepsilon$.

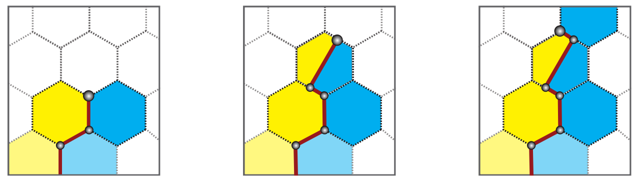

Figure 3: "Multistep" procedure by which the exploration process gets through a mixed hexagon.

We state without proof some properties of our exploration process.

Proposition 2.4. *Let $\gamma_{[0,t]}^\varepsilon$ be the line segments formed by the process up till time t, and $\Gamma_{[0,t]}^\varepsilon$ the hexagons revealed by the exploration process. Let $\partial_\varepsilon \Omega_\varepsilon^t = \partial_\varepsilon \Omega_\varepsilon \cup \Gamma_{[0,t]}^\varepsilon$ and let $\Omega_\varepsilon^t = \Omega_\varepsilon \setminus \Gamma_{[0,t]}^\varepsilon$. Then, the quadruple $(\Omega_\varepsilon^t, \partial_\varepsilon \Omega_\varepsilon^t, \mathbb{X}_t^\varepsilon, b_\varepsilon)$ is admissible. Furthermore, the exploration process in Ω_ε^t from $\mathbb{X}_t^\varepsilon$ to b_ε has the same law as the original exploration process from a_ε to b_ε in Ω_ε conditioned on $\Gamma_{[0,t]}^\varepsilon$.*

2.4 Statement of the Main Theorem and Lemmas

The principal result of this note may be stated as follows:

Main Theorem. *Let Ω be a domain with $M(\partial\Omega) < 2$. Let a and b be two prime ends at the boundary of the domain. Let μ_ε be the probability measure on random curves inherited from the exploration process of the percolation problem described in §2.1 with lattice spacing ε. Then*

$$\mu_\varepsilon \underset{\mathcal{L}}{\Longrightarrow} \mu_0,$$

where μ_0 has the law of chordal SLE_6 from a to b.

We will need the following auxiliary results:

Lemma 2.5 (Tightness)**.** *Let μ' be any limit point, in the weak* Hausdorff topology on compact sets, of μ_ε. Then μ' gives full measure to Löwner curves in Ω from a to b.*

Corollary/Remark. *As a residual consequence of the main theorem and Lemma 2.5, we obtain the following: With probability one, chordal SLE_6 in an arbitrary domain Ω with $M(\partial\Omega) < 2$ is, in fact, a curve.*

Furthermore,

Lemma 2.6 (Admissibility)**.** *The limit point μ' gives full measure to curves with upper Minkowski dimension less than $2 - \psi'$ for some $\psi' > 0$.*

And finally,

Lemma 2.7 (Cardy's Formula)**.** *Let (Ω, a, b, c, d) be a conformal rectangle – that is to say, a domain with boundary prime ends a, b, c, d, listed in counter-clockwise order. Assume that $M(\partial\Omega) < 2$. Let $C_\varepsilon(\Omega, a, b, c, d)$ denote the probability that there exists a blue crossing from $[a, b]$ to $[c, d]$ on the ε-lattice of Ω. Consider the (unique) conformal map which takes (Ω, a, b, c, d) to $(\mathbb{H}, 1 - x, 1, \infty, 0)$, where, clearly, $0 < x < 1$ and $x = x(\Omega, a, b, c, d)$. Then*

$$\lim_{\varepsilon \to 0} C_\varepsilon(\Omega, a, b, c, d) = F(x) := \frac{\int_0^x (s(1-s))^{-2/3}\, ds}{\int_0^1 (s(1-s))^{-2/3}\, ds} \tag{1}$$

In Lemma 2.5 (and 2.6), a stronger notion of convergence is available. Indeed, for domains which are regular enough, the results of [1] provide weak convergence in the distance provided by the sup–norm:

$$\mathrm{dist}(\gamma_1, \gamma_2) = \inf_{\varphi_1, \varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|,$$

where the infimum is over all possible parametrizations. For our purposes – where prime ends are a concern – we will consider a weighted sum of the distances within various regions between the curves. The construction is standard and precise details will be spelled out in Subsection 3.3. We will denote the appropriate distance by **Dist** (see Definition 3.10).

Lemma 2.8 (Restricted Uniform Continuity)**.** *Let Ω be a domain with $M(\partial\Omega) < 2$ and a, b, c, d be four points or prime ends on $\partial\Omega$. Let $\vartheta, \Delta > 0$ and consider curves which start at a and end at c. Then there exists a set $\Xi_{\vartheta,\Delta}$ of such curves and $\eta > 0$, such that if $\gamma_1 \in \Xi_{\vartheta,\Delta}$, $M(\gamma_2) < 2$, $M(\gamma_1) < 2$, and* $\mathbf{Dist}(\gamma_1, \gamma_2) < \eta$*, then $\forall T \geq 0$ such that $\gamma_1([0, T])$ and $\gamma_2([0, T])$ do not visit the Δ neighborhood of c, provided that b, c, d are at the boundaries of both domains $\Omega \setminus \gamma_1((0, T])$ and $\Omega \setminus \gamma_2((0, T])$,*

$$|C_\varepsilon(\Omega \setminus \gamma_1([0, T]), \gamma_1(T), b, c, d) - C_\varepsilon(\Omega \setminus \gamma_2([0, T]), \gamma_2(T), b, c, d)| < \frac{1}{2}\vartheta$$

and for all ε sufficiently small,

$$\mu_\varepsilon(\Xi_{\vartheta,\Delta}) > 1 - \frac{1}{2}\vartheta,$$

with the same for μ'.

2.5 Proof of the Main Theorem

Let us show how to derive our Main Theorem from the preceding lemmas. We closely follow the strategic initiative outlined in the expositions of [11] and [12].

Fix Ω with $M(\partial\Omega) < 2$ and two boundary prime ends a and c. Note that here we denote by c (instead of b) the finishing point of the Exploration Process. Let us note that the collection of measures (μ_ε) defined by the Exploration Process on ε-lattice is weakly precompact as a set of regular measures defined on the space of compact subsets of $\mathrm{Clos}(\Omega)$ with the Hausdorff metric. Thus to prove the Main Theorem it is enough to show that any weak limit point μ', of μ_ε, has the law of SLE_6 from a to c in Ω.

By Lemma 2.5, μ' gives full measure to Löwner curves. Let w_t be the random driving function of the curve. To finish the proof, we need to show that w_t has the law of B_{6t}, where B_t is the standard one dimensional Brownian Motion started at 0.

We will use the following a priori estimate on w_t.

Lemma 2.9 (*A priori* estimate)**.**

$$\mathbb{P}[w_t > n] \leq C_1 \exp\left(-C_2 \frac{n}{\sqrt{t}}\right),$$

for some absolute constants C_1 and C_2.

The lemma implies, in particular, that all moments of w_t are finite. We postpone the proof of this lemma until Section 3.

Let us add two boundary prime ends b and d so that (a, b, c, d) are listed counterclockwise and give the discrete Exploration Process from a to c, $\mathbb{X}^\varepsilon_t$, the Löwner parametrization. Now fix $t > 0$. By definition, the faces on the right side of the Exploration Process are blue, and the faces on the left side are yellow. A blue crossing from $[a, b]$ to $[c, d]$ can either touch the exploration path $\mathbb{X}^\varepsilon_{[0,t]}$, or avoid it. In any case, it produces a blue crossing between $[\mathbb{X}^\varepsilon_t, b]$ to $[c, d]$ in $\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}$. And vice versa, any blue crossing between $[\mathbb{X}^\varepsilon_t, b]$ to $[c, d]$ in $\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}$ produces a blue crossing from $[a, b]$ to $[c, d]$ in Ω.

This means that we can write the following Markov identity for the crossing probabilities

$$C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right) = C_\varepsilon\left(\Omega, a, b, c, d \mid \mathbb{X}^\varepsilon_{[0,t]}\right), \tag{2}$$

so

$$\mathbb{E}_{\mu_\varepsilon}\left[C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right)\right] = C_\varepsilon\left(\Omega, a, b, c, d\right). \tag{3}$$

Now let us fix $t > s > 0$. Then by the same reasoning we have

$$\mathbb{E}_{\mu_\varepsilon}\left[C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right) \mid \mathbb{X}^\varepsilon_{[0,s]}\right] = C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,s]}, \mathbb{X}^\varepsilon_s, b, c, d\right). \tag{4}$$

We now pass to the limit as $\varepsilon \to 0$. Let μ' be any limiting point of the measures μ_ε. Consider some $\mathbb{X}_{[0,t]} \in \Xi_{\vartheta,\Delta}$ (see statement of Lemma 2.8) and its Löwner parametrization η–neighborhood $\mathcal{N}_{\eta,t}$. Note that $\mathcal{N}_{\eta,t}$ is certainly contained in the sup–norm η neighborhood, and in fact has positive measure: by examining the inverse image of the η neighborhood after a conformal map, we see that it contains a small sup–norm neighborhood. Let Ω_t be the connected component of $\Omega \setminus \mathbb{X}_{[0,t]}$ such that $c \in \partial\Omega_t$. By Lemma 2.6, we can assume that $M(\mathbb{X}) < 2$.

The capacitance at c of the curve $\mathbb{X}_{[0,t]}$ is $2t$ so either the curve stays at some distance Δ away from c, or it has to pass through a fixed rectangle of large conformal modulus, and hence for small enough Δ, $\mathbb{X}_{[0,t]} \in \Xi_{\vartheta,\Delta}$, with large probability. This means that Lemma 2.8 applies and hence if η and ε are small enough and $\mathbb{X}^\varepsilon_{[0,s]} \in \mathcal{N}_{\eta,s}$, then $C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,s]}, \mathbb{X}^\varepsilon_s, b, c, d\right)$ is close to $C_\varepsilon\left(\Omega_s, \mathbb{X}_s, b, c, d\right)$. A second application of Lemma 2.8 implies that for any $\mathbb{X}^\varepsilon_{[0,t]} \in \mathcal{N}_{\eta,t}$, the value of $C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right)$ is close to $C_\varepsilon\left(\Omega_t, \mathbb{X}_t, b, c, d\right)$, provided that ε and η are small enough *and provided that b and d are still in $\partial\Omega_t$*. Averaging over all possible continuations of $\mathbb{X}_{[0,s]}$ and taking small enough η and ε, we get from equation (4)

$$\begin{aligned}\Big|C_\varepsilon\left(\Omega_s, \mathbb{X}_s, b, c, d \mid b, d \in \partial\Omega_s\right) - \mathbb{E}_{\mu_\varepsilon} C_\varepsilon\left(\Omega_t, \mathbb{X}_t, b, c, d \mid \mathbb{X}_{[0,s]},\ b, d \in \partial\Omega_t\right)\Big| \\ \le \mathbb{P}_{\mu_\varepsilon}(b \notin \partial\Omega_t \text{ or } d \notin \partial\Omega_t) + \delta \quad (5)\end{aligned}$$

for an arbitrarily small δ.

By Lemma 2.7, $\lim_{\varepsilon\to 0} C_\varepsilon\left(\Omega_s, \mathbb{X}_s, b, c, d\right) = C_0\left(\Omega_s, \mathbb{X}_s, b, c, d\right)$. If we now let $\varepsilon \to 0$ and $\eta \to 0$ in (5) and use the bounded convergence theorem, we get the approximate Markov property for the measure μ'

$$\begin{aligned}\Big|C_0\left(\Omega_s, \mathbb{X}_s, b, c, d \mid b, d \in \partial\Omega_s\right) - \mathbb{E}_{\mu'} C_0\left(\Omega_t, \mathbb{X}_t, b, c, d \mid \mathbb{X}_{[0,s]},\ b, d \in \partial\Omega_t\right)\Big| \\ \le \mathbb{P}_{\mu'}(b \notin \partial\Omega_t \text{ or } d \notin \partial\Omega_t). \quad (6)\end{aligned}$$

Notice that the map

$$h_t(z) = \frac{g_t(z) - g_t(d)}{g_t(b) - g_t(d)},$$

where $g_t(z)$ is the Löwner map, maps the rectangle $(\Omega_t, \mathbb{X}_t, b, c, d)$ conformally onto

$$\left(\mathbb{H}, \frac{w_t - g_t(d)}{g_t(b) - g_t(d)}, 1, \infty, 0\right).$$

By Cardy's identity (Lemma 2.7),

$$C_0(\Omega_t, \mathbb{X}_t, b, c, d) = F\left(\frac{g_t(b) - w_t}{g_t(b) - g_t(d)}\right). \quad (7)$$

Using equation (7), we can rewrite the equation (6) as

$$\left| F\left(\frac{g_s(b)-w_s}{g_s(b)-g_s(d)}\right) - \mathbb{E}_{\mathbb{X}_{[s,t]}}\left(F\left(\frac{g_t(b)-w_t}{g_t(b)-g_t(d)}|\mathbb{X}_{[0,s]}\right) \mid b,d\in\partial\Omega_t\right)\right| \\ \leq \mathbb{P}(b\notin\partial\Omega_t \text{ or } d\notin\partial\Omega_t). \quad (8)$$

Let us now define the flows $b_t := g_t(b) - w_t$, $d_t := g_t(d) - w_t$. We also put $b_t = 0$ if $b \notin \partial\Omega_t$ and $d_t = 0$ if $d \notin \partial\Omega_t$. Thus $b_0 = g_0(b) > 0$, $d_0 = g_0(d) < 0$. Now let us start moving b and d closer to c. We will move them in such a manner that $d_0 = -2b_0$. By the Löwner parametrization in the half plane, $g_t(g_0^{-1}(z)) = z + 2t/z + \mathbb{O}(1/z^2)$ for $z \to \infty$.

Let us first provide the expansion of the first term of the left hand side of (8), when b_0 is close to infinity.

$$\begin{aligned} F\left(\frac{g_s(b)-w_s}{g_s(b)-g_s(d)}\right) &= F\left(\frac{b_0 - w_s + 2s/b_0 + \mathbf{O}(1/b_0^2)}{b_0 + 2s/b_0 + \mathbf{O}(1/b_0^2) + 2b_0 + s/b_0 + \mathbf{O}(1/b_0^2)}\right) \\ &= F(1/3) - w_s/3F'(1/3)1/b_0 + \left(s/3F'(1/3) + w_s^2/12F''(1/3)\right)1/b_0^2 + \mathbf{O}((w_s/b_0)^3) \\ &= A + \frac{Bw_s}{b_0} + \frac{C(w_s^2-6s)}{b_0^2} + \mathbf{O}((w_s/b_0)^3) \end{aligned} \quad (9)$$

for some A, B, and C.

By the same reasoning, the second expression on the left hand side of (8) is equal to

$$\mathbb{E}_{\mathbb{X}_{[s,t]}}\left(F\left(\frac{g_t(b)-w_t}{g_t(b)-g_t(d)}|\mathbb{X}_{[0,s]}\right) \mid b,d\in\partial\Omega_t\right) \\ = A + \frac{B\mathbb{E}(w_t|w_s)}{b_0} + \frac{C\mathbb{E}((w_t^2-6t)|w_s)}{b_0^2} + \mathbf{O}(|\mathbb{E}[w_t/b_0^3]) \quad (10)$$

The last expression of (8) decays exponentially with b_0. Indeed, assume that $b \notin \partial\Omega_t$ (the case $d \notin \partial\Omega_t$ is analogues). The boundedness of the derivative at infinity of the map $g_t(g_0^{-1}(z))$ forces the image of the exploration process up to time t under the map g_0 to stay inside the box $[0, b_0] \times [0, 2\sqrt{t}]$. By Cardy's formula, such a probability decays exponentially with b_0.

We can compare the coefficients of $1/b_0$ and $1/b_0^2$ of the two terms on the left hand side equation (8) with the corresponding coefficients in equations (9) and (10) to get

$$\mathbb{E}(w_t|w_s) = w_s, \quad \mathbb{E}(w_t^2 - 6t|w_s) = w_s^2 - 6s. \quad (11)$$

Therefore both w_t and $w_t^2 - 6t$ are continuous martingales, which, by the usual characterization of Brownian Motion, implies that w_t has the law of B_{6t}. □

3 Technical Lemmas and Proofs

3.1 A Restricted BK–Inequality

Here we will prove an inequality that will be needed for proofs in several other places.

Suppose A and B are two events. Then the BK inequality [2] states that (for suitable probability spaces) the probability of the *disjoint* occurrence of A and B is bounded above by the product of their probabilities. The most general version of this is Reimer's inequality [9] (see also [3] for more background and a self–contained proof), which holds for arbitrary product probability spaces. For the model at hand, we do not have a product probability space; Reimer's inequality would, in the present context, yield the desired result only for *flower* disjoint events. Unfortunately, we have need of a stronger statement; specifically, for disjoint path–type events where the individual paths may use the same flower. In fact, as the following example demonstrates, a general BK inequality does not hold in our system. However, as we later show, an abridged version holds for path–type events.

Example 3.1. Let A be the event of a blue connection between petals 1, 4, and 5 (without any requirement on the color of the petals 1, 4, and 5), and let $B = \{\text{petals 1, 4, 5 are blue}\}$. Observe that B and B^c are defined entirely on the petals 1, 4, 5, whereas A is defined on the complementary set. Therefore we have $A \cap B^c = A \circ B^c$. By Example 6.1 of [5], we know that $\mathbb{P}(A \cap B) < \mathbb{P}(A)\mathbb{P}(B)$. But this immediately implies that $\mathbb{P}(A \circ B^c) > \mathbb{P}(A)\mathbb{P}(B^c)$.

Before tending to the detailed analysis of flowers, let us first introduce the notion of disjoint occurrence for non–negative random variables.

Definition 3.2. Let $a_i, b_j \geq 0$ and let

$$X = \sum_1^n a_i \mathbf{1}_{A_i}, \quad Y = \sum_1^m b_j \mathbf{1}_{B_j},$$

where $A_i \cap A_k = \emptyset$ for $i \neq k$ and $B_j \cap B_l = \emptyset$ for $j \neq l$. We define

$$X \circ Y = \sum_{i,j} a_i b_j \mathbf{1}_{A_i \circ B_j}.$$

If the usual BK inequality holds then linearity immediately gives

$$\mathbb{E}(X \circ Y) \leq \mathbb{E}(X)\mathbb{E}(Y).$$

We will be working with this slight generalization; what we have in mind is the hexagon disjoint occurrence of paths, and in the case of paths of different colors, sharing of the iris may occur. To be precise, we have the following definition:

Definition 3.3. Let $\Lambda_{\mathfrak{F}}$ denote a flower arrangement and let S and T denote sets in $\Lambda_{\mathfrak{F}}$ which contain no irises. Let $X^b_{S,T}$ denote the indicator of the event that all hexagons in S and T are blue and that there is a blue path – possibly including irises – connecting S and T. Similarly we define $X^y_{S,T}$ to be the yellow version of this event. Now if S' and T' are two other sets of $\Lambda_{\mathfrak{F}}$ which are disjoint from S and T and also do not contain irises, then we may define $X^b_{S,T} \circ X^b_{S',T'}$ in accord with the usual fashion. However, for present purposes, in the event corresponding to $X^b_{S,T} \circ X^y_{S',T'}$, the two paths may share a mixed iris.

Lemma 3.4. *Let $X^{\ell_1}_{S_1,T_1}, X^{\ell_2}_{S_2,T_2}, \ldots, X^{\ell_n}_{S_n,T_n}$ be the indicator functions of path–type events as described in Definition 3.3, where $\ell_i \in b, y$, then*

$$\mathbb{P}(X^{\ell_1}_{S_1,T_1} \circ X^{\ell_2}_{S_2,T_2} \circ \ldots, X^{\ell_n}_{S_n,T_n}) \leq \mathbb{P}(X^{\ell_1}_{S_1,T_1})\mathbb{P}(X^{\ell_2}_{S_2,T_2}) \ldots \mathbb{P}(X^{\ell_n}_{S_n,T_n}).$$

Proof. Our proof is slightly reminiscent of the proof of Lemma 6.2 in [5]. Let σ denote a configuration of petals and filler and let I denote a configuration of irises. We will use induction; first we prove the statement for the case of exactly one flower (i.e. supposing there is only one flower in all of $\Lambda_{\mathfrak{F}}$) and two paths, whose indicator functions we denote by X and Y. We write

$$\mathbb{E}(X \circ Y) = \mathbb{E}_\sigma[\mathbb{E}_I(X \circ Y|\sigma)].$$

If we can show that $\mathbb{E}_I(X \circ Y|\sigma) \leq \mathbb{E}_I(X|\sigma) \circ \mathbb{E}_I(Y|\sigma)$, then we may apply the BK–inequality to the outer expectation to yield the desired result since, on the outside, the measure is independent. It is clear that the function $\mathbb{E}(X \circ Y|\sigma)$ can only take on five different values; we write

$$\begin{aligned} \mathbb{E}(X \circ Y|\sigma) = 1 \cdot \mathbf{1}_{\mathcal{O}(X \circ Y)}(\sigma) \\ + (a+s) \cdot \mathbf{1}_{A_1(X \circ Y)}(\sigma) \\ + (1/2) \cdot \mathbf{1}_{A_2(X \circ Y)}(\sigma) \\ + (a+2s) \cdot \mathbf{1}_{A_3(X \circ Y)}(\sigma) \\ + s \cdot \mathbf{1}_{\mathcal{F}(X \circ Y)}(\sigma), \end{aligned} \tag{12}$$

where e.g.

$$\mathcal{O}(X \circ Y) = \{\sigma \mid \mathbb{E}(X \circ Y|\sigma) = 1\}.$$

It is not difficult to see that $\mathcal{O}(X \circ Y)$ is the set of σ configurations where $X \circ Y$ has occurred on the complement of the iris. The remaining terms warrant some discussion. We first point out that these terms correspond to configurations where the flower is pivotal for the achievement of at least one of X and Y, and, due to the nature of the events in question, petal arrangements in these configurations satisfy certain constraints. For instance, configurations in A_3 must exhibit a petal arrangement such that one of the paths is in a position where it must transmit through the iris, which can be accomplished by the preferred color or two of the split configurations; the flower must *not* be in a triggering configuration and, needless to say, the other path has already occurred (independent of the iris).

Finally we observe that $\sigma \in \mathcal{F}(X \circ Y)$ implies that both paths must use the iris and therefore can only occur when the paths in question have different colors. It is not hard to see, via petal counting, that $\mathcal{F}(X \circ Y)$ forces the alternating configuration of petals and that indeed, we have a situation of a "parallel transmission" through the iris, with exactly one iris configuration which achieves both desired transmissions. We also note that in similar expressions for $\mathbb{E}(X|\sigma)$ and $\mathbb{E}(Y|\sigma)$, the corresponding terms $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ will be empty, since e.g. if the path is blue and some iris is capable of achieving the transmission, then certainly the pure blue iris will achieve the transmission.

Let us expand $\mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma)$ in the sense defined above:

$$\begin{aligned} \mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma) = 1 &\cdot \mathbf{1}_{\mathcal{O}(X)\circ\mathcal{O}(Y)}(\sigma) \\ &+ (a+s) \cdot [\mathbf{1}_{\mathcal{O}(X)\circ A_1(Y)}(\sigma) + \mathbf{1}_{A_1(X)\circ\mathcal{O}(Y)}(\sigma)] \\ &+ (1/2) \cdot [\mathbf{1}_{\mathcal{O}(X)\circ A_2(Y)}(\sigma) + \mathbf{1}_{A_2(X)\circ\mathcal{O}(Y)}(\sigma)] \\ &+ (a+2s) \cdot [\mathbf{1}_{\mathcal{O}(X)\circ A_3(Y)}(\sigma) + \mathbf{1}_{A_3(X)\circ\mathcal{O}(Y)}(\sigma)] \\ &+ (a+s)^2 \cdot [\mathbf{1}_{A_1(X)\circ A_1(Y)}(\sigma)] \\ &+ \mathcal{R}(a,s,\sigma), \end{aligned} \tag{13}$$

where $\mathcal{R}(a,s,\sigma)$ contains all the remaining terms in the expansion, e.g. the terms

$$(1/2)(a+s) \cdot [\mathbf{1}_{A_1(X)\circ A_2(Y)}(\sigma) + \mathbf{1}_{A_2(X)\circ A_1(Y)}(\sigma)] \tag{14}$$

and

$$(a+s)(a+2s) \cdot [\mathbf{1}_{A_1(X)\circ A_3(Y)}(\sigma) + \mathbf{1}_{A_3(X)\circ A_1(Y)}(\sigma)]. \tag{15}$$

We claim that (14) will evaluate to zero for each σ: In the first term, $A_1(X)$ requires that the petals exhibit a configuration which precludes a trigger and $A_2(Y)$ requires the petals to exhibit a configuration which leads to a trigger, and similarly for the second term. The terms in (15) may or may not evaluate to zero for all σ *a priori*, but in any case will not be needed.

Now we match up the terms in (12) and (13) and demonstrate that indeed $\mathbb{E}(X \circ Y|\sigma) \leq \mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma)$. First note that $\mathcal{O}(X \circ Y) = \mathcal{O}(X) \circ \mathcal{O}(Y)$. Next, as discussed previously, we see that $A_i(X \circ Y) \subset (A_i(X) \circ \mathcal{O}(Y)) \cup (\mathcal{O}(X) \circ A_i(Y)), 1 \leq i \leq 3$. Finally, and this is the key case, we claim that $\mathcal{F}(X \circ Y) \subset A_1(X) \circ A_1(Y)$. This follows from the observation we made before, which is that if $\sigma \in \mathcal{F}(X \circ Y)$, then we must see the alternating configuration on the flower, requiring next to nearest neighbor transmissions through the iris for both paths; such a σ certainly lies in $A_1(X) \circ A_1(Y)$. Thus we are done, assuming that $(a+s)^2 \geq s$ – but this is equivalent to the statement that $a^2 \geq 2s^2$.

We have established the claim for the case of a single flower and two paths. Next we may induct on the number of flowers, as follows. Suppose now the claim is established for $K-1$ flowers. We can now let σ denote the configuration of all petals, filler, and irises of the first $K-1$ flowers. We condition on σ as above and adapt the notation so that the sets $\mathcal{O}$, A_i's, and $\mathcal{F}$ correspond to the K^{th} flower. The argument can then be carried out exactly as above to yield the result for K flowers and two paths. Finally we induct on the number of paths. Suppose the claim is true for $n-1$ paths. Since the $\circ$ operation is associative, we consider $(X_1 \circ \cdots \circ X_{n-1}) \circ X_n$, where the X_i's are indicator functions of the n paths. We simply view $(X_1 \circ \cdots \circ X_{n-1})$ as a single path–type event and repeat the proof (note that the analogue of equation (13) may now contain non–trivial $\mathcal{F}$–type terms; these are immaterial since what is listed is already enough for an upper bound). This argument is sufficient since no more than two paths may share an iris under any circumstance. □

3.2 Generalization of Cardy's Formula for $M(\partial\Omega) < 2$

Here we provide a proof of Lemma 2.7. As described in §2.1, [5] contains a proof of Cardy's formula for piecewise smooth domains, so what is needed here is a generalization to domains Ω with $M(\partial\Omega) < 2$. What we will prove is the following:

Lemma 3.5. *Let Ω denote any conformal triangular domain with $M(\partial\Omega) < 2$. Let U_ε^Y, V_ε^Y and W_ε^Y denote the crossing probability functions as defined in Ω for the lattice at scale ε. Then for the model as defined in §2.1, we have*

$$\lim_{\varepsilon \to 0} U_\varepsilon^Y = h_{\mathcal{C}},$$

with similar results for V_ε^Y and W_ε^Y and the corresponding blue versions of these functions, where $h_{\mathcal{A}}$, $h_{\mathcal{B}}$ and $h_{\mathcal{C}}$ are the Cardy–Carleson functions.

To prove the current statement, we start by repeating the proof in [5] up to Lemma 7.2 and Corollary 7.4 – the one place where the assumption on a piecewise smooth boundary is used. We now give a quick exposition of the (relevant portions of the) strategy of proof in [5]. The idea (directly inherited from [12]) is to represent the derivative of the crossing probability functions as a "three–arm" event, e.g. two blue paths and one yellow path from some point to the boundaries, with all paths disjoint, and then derive Cauchy–Riemann type identities by switching the color of one of the arms.

In order to accomplish this color switching in our model, it was necessary to introduce a *stochastic* notion of disjointness. This amounted to the introduction of a large class of random variables which indicate whether or not a percolation configuration contributes to the event of interest (e.g. a blue path from $\mathcal{A}$ to $\mathcal{B}$, separating z from $\mathcal{C}$). We call the restrictions and permissions given by these random variables $*$–*rules*. The $*$–rules may at times call a self–avoiding path illegitimate if it contains *close encounters*, i.e. comes within one unit of itself; on the other hand, the $*$–rules may at other times permit a path which is not self–avoiding but in fact shares a hexagon. Thus the $*$–rules are invoked only at shared hexagons and close encounter points of a path. When a close encounter or sharing at a hexagon is required to achieve the desired path event it is called an *essential lasso point*.

The fact that these $*$–rules may be implemented by random variables in a fashion which allows color switching is the content of Lemma 3.17 in [5]. The strategy was then to first prove that the $*$–version of e.g. the function U_ε, denoted U_ε^*, converges to $h_{\mathcal{C}}$, then show that in the limit the starred and unstarred versions of the function coincide. For the current work, the precise statement is as follows:

Lemma 3.6. *Let* Ω *be a domain such that*

$$M \equiv M(\partial\Omega) < 2.$$

Let z *denote a point in* Ω*. Consider the (blue version of the) function* $U_\varepsilon(z)$ *as defined in §2.1. Let* $U_\varepsilon^*(z)$ *denote the version of* U_ε *with the* $*$*-rules enforced. Then,*

$$\lim_{\varepsilon\to\infty} |U_\varepsilon^*(z) - U_\varepsilon(z)| = 0.$$

In particular, on closed subsets of Ω*, the above is uniformly bounded by a constant times a power of* ε*.*

Before we begin the proof we need some standard percolation notation.

Definition 3.7. Back on the unit hexagon lattice, if L is a positive integer, let B_L denote a box of side L centered at the origin. Further, let $\Pi_5(L)$ denotes the event of five disjoint paths, not all of the same color, starting from the origin and ending on ∂B_L. Now let $m < n$ be positive integers, and let $\Pi(n,m)$ denote the event of five long arms, not all of the same color, connecting ∂B_m and ∂B_n. We use the notation $\pi_5(n)$ and $\pi_5(n,m)$ for the probabilities of $\Pi_5(n)$ and $\Pi_5(n,m)$, respectively.

Proof of Lemma 3.6. We set $N = \varepsilon^{-1}$ and, without apology, we will denote the relevant functions by U_N. For convenience we recap the proof of Lemma 7.2 in [5] (with one minor modification). Let us first consider the event which is contained in both the starred and unstarred versions of the u–function, namely the event of a self–avoiding, non–self–touching path separating z from $\mathcal{C}$, etc. We will denote the indicator function of this event by $\mathcal{U}_N^-$. Similarly, let us define an event, whose indicator is $\mathcal{U}_N^{*+}$, that contains both the starred and unstarred versions: this is the event that a separating path of the required type exists, with no restrictions on self–touching, and is allowed to share hexagons provided that permissions are granted. It is obvious that

$$\mathbb{E}[\mathcal{U}_N^{*+} - \mathcal{U}_N^-] \geq |U_N^* - U_N|. \tag{16}$$

We turn to a description of the configurations, technically on (ω, X) (the enlarged probability space which include the permissions), for which $\mathcal{U}_N^{*+} = 1$ while $\mathcal{U}_N^- = 0$. In such a configuration, the only separating paths contain an *essential* lasso point which, we remind the reader, could be either a shared hexagon or a closed encounter pair. Let us specify the lasso point under study to be the last such point on the journey from $\mathcal{A}$ to $\mathcal{B}$ (i.e. immediately after leaving this point, the path must capture z without any further sharing or self–touching, then return to this point and continue on to $\mathcal{B}$). For standing notation, we denote this "point" by z_0. A variety of paths converge at z_0: certainly there is a blue path from $\mathcal{A}$, denoted $B_{\mathcal{A}}$, a blue path to $\mathcal{B}$, denoted $B_{\mathcal{B}}$, and an additional loop starting from z_0 (or its immediate vicinity) which contains z in its interior. The loop we may view as two blue paths of comparable lengths, denoted L_z^1 and L_z^2. However, since the lasso point was deemed to be essential, there are two additional yellow arms emanating from the immediate vicinity of z_0. These yellow arms may themselves encircle the blue loop and/or terminate at boundary $\mathcal{C}$. We denote these yellow paths $Y_{\mathcal{C}}^1$ and $Y_{\mathcal{C}}^2$.

Since z_0 is the last lasso point on the blue journey from $\mathcal{A}$ to $\mathcal{B}$, we automatically get that the two loop arms are *strictly* self–avoiding. Also, without loss of generality, we may take the yellow arms to be strictly self–avoiding. Further, by Lemma 4.3 of [5], we may take either the portion of the path from $\mathcal{A}$ to z_0 to be strictly self–avoiding or the portion of the path from $\mathcal{B}$ to z_0 to be strictly self–avoiding. To summarize, we have six paths emanating from z_0, four blue and two yellow, with all paths disjoint except for possible sharings between $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$. For simplicity, let us start with the connected component of z in $\Omega \setminus (\alpha_k \cup \beta_k \cup \gamma_k)$ where $\alpha_k, \dots, \gamma_k$ are short crosscuts defining the prime ends $e_{AB}, \dots, e_{CA}$, respectively. It is noted that in this restricted setting, the various portions of the boundary are at a finite (macroscopic) distance from one another. Thus, on a mesoscopic scale, we are always near only a single boundary.

The case where z_0 is close to z is handled by RSW-type bounds (see proof of Lemma 7.2 in [5]). The terms where z_0 is in the interior follow from the 5^+ arms; these arguments are the subject of Lemma 7.2 and Lemma 7.3 in [5]. We are left with the case where say z_0 is within a distance of N^λ of the boundary but outside some box of side N^{μ_2} separating e_{AB} from z.

Let $\delta > 0$. For N large enough, $\partial\Omega$ can be covered by no more than $J_\delta N^{M+\delta-\lambda}$ boxes of side N^λ. Now we take these boxes and expand by a factor of, say, two and we see that the region within N^λ of the boundary can be covered by $J_\delta N^{M+\delta-\lambda}$ boxes of side $2N^\lambda$. We surround each of these boxes by a box of side N^{μ_1}, where $\mu_2 > \mu_1 > \lambda$.

Now suppose z_0 is inside the inner box. We still have the six arms $B_{\mathcal{A}}$, $B_{\mathcal{B}}$, L_z^1, L_z^2, $Y_{\mathcal{C}}^1$ and $Y_{\mathcal{C}}^2$, but since z_0 is now close to some boundary, we expect some arm(s) to be short (i.e. shorter than N^λ). We note that the box of side μ_1 is still away from e_{AB}, and therefore we cannot have more than one of $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$ be short due to being close to the boundary. Also, since z must be a distance of order N away from the boundary, z is outside of both of these boxes and therefore both L_z^1 and L_z^2 are long. The upshot is that regardless of which boundary z_0 is close to, one and only one of the six arms will be short: If z_0 is close to $\mathcal{A}$ (respectively $\mathcal{B}$), then $B_{\mathcal{A}}$ (respectively $B_{\mathcal{B}}$) will be short, and if z_0 is close to $\mathcal{C}$, then a moment's reflection will show that only one of the yellow arms will be short.

What we have is then five long arms and one short arm emanating from the immediate vicinity of z_0, and these arms either end on some boundary or the boundary of the outer box of side N^{μ_1}. For reasons which will momentarily become clear, we will now perform a color switch. Topologically, the two yellow arms separate L_z^1 and L_z^2 from $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$. Denote the outer box by B_{μ_1} and consider now the region $T \equiv \Omega \cap B_{\mu_1}$. The two yellow arms together form a "crosscut" (in the sense of Kesten [7]) of T . This crosscut separates T into two disjoint regions T_b and T_l, where T_b contains $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$ and T_l contains L_z^1 and L_z^2. We condition on the crosscut which minimizes the area of T_l. Next we apply Lemma 4.3 of [5] to reduce the blue arm adjacent to the longer of the two yellow arms – which we take to be $Y_{\mathcal{C}}^1$ – to be strictly self–avoiding, which without loss of generality we assume to be $B_{\mathcal{A}}$. Since $B_{\mathcal{A}}$ forms a crosscut of T_b, there is a crosscut which maximizes the region which contains $B_{\mathcal{B}}$, which we denote Ω_B. The region Ω_B is now an unconditioned region, and we may apply Lemma 3.17 of [5] to switch the color of $B_{\mathcal{B}}$ from blue to yellow, while preserving the probability. The resulting yellow path we will denote $Y_{\mathcal{B}}$.

We now have three blue paths and three yellow paths. The blue paths are now all strictly self–avoiding. $Y_{\mathcal{C}}^1$ is still strictly self–avoiding, but the path $Y_{\mathcal{B}}$ may very well interact with (i.e. share hexagons with, due to the $*$-rules) $Y_{\mathcal{C}}^2$. If indeed there is sharing, then let $\hat{Y} = Y_{\mathcal{B}} \cup Y_{\mathcal{C}}^2$ be the geometric union of the two paths. $\hat{Y}$ can then be reduced to be a strictly self–avoiding path, which we now denote Y. In any case, we now have (at least) five long paths emanating from z_0, three blue and two yellow, with the yellow paths separating the blue paths, and with all paths strictly self–avoiding. The probability of such an event is certainly bounded above (possibly strictly since the boxes will most likely intersect Λ^c) by the full space event $\Pi_5(N^{\mu_1}, 2N^{\lambda})$ – see Definition 3.7. The upshot of Lemma 5 of [7] is that

$$\pi_5(N^{\mu_1}, 2N^{\lambda}) \leq C \left(\frac{N^{\lambda}}{N^{\mu_1}} \right)^2, \tag{17}$$

where C is a constant. This result can, almost without apology, be taken verbatim from [7]; the proviso concerned "relocation of arms" which was discussed in the first paragraph of the proof of Lemma 7.3 in [5]. We consider (17) to be established.

If we sum over all such boxes of side $2N^{\lambda}$, we find that the contribution from the near boundary regions is a constant times

$$N^{M+\delta-\lambda+2\lambda-2\mu_1} = N^{M+\delta+\lambda-2\mu_1}.$$

Since $M < 2$, we may first choose δ and λ such that $M + \delta + \lambda < 2$, then we may first choose μ_2 and then μ_1 large enough so that the exponent is negative. Finally let us take care of the crosscuts. For k large, the event that a path emanates from the crosscut e.g. β_k

and goes to $\mathcal{B}$ tends to 0 as $k \to \infty$ (uniformly in N for all N sufficiently large): Indeed, a path emanating from β_k must pass through a minimal sized "bottleneck" – whose diameter, η_k, tends to 0 as $k \to \infty$. This implies the existence of a long path emanating from a small region, with probability which vanishes with some power of η_k. Similarly for the other two prime ends. All estimates are uniform in z provided z remains a fixed non–zero (Euclidean) distance from the boundary.

The proof of Lemma 3.6 for V_N and W_N are the same. □

Corollary 7.4 of [5] concerned the difference between the blue and yellow versions of these functions (Cauchy–Riemann relations are only established for color–neutral sums). However, the argument of Corollary 7.4 in [5] reduced the difference between the two colored versions to six arm events in the bulk and five arm events near the boundary, to which the above arguments can be applied. Replacing Lemma 7.2 (and Lemma 7.3) in [5] with Lemma 3.6 gives a proof of Lemma 3.5.

Now we are in position to prove Lemma 2.7.

Proof of Lemma 2.7. By conformal invariance of the limiting functions (and an application of the Schwarz–Christoffel transform), it is a well established fact that limiting behavior of the three Carleson–Cardy functions implies Cardy's Formula. While we actually establish convergence of the lattice functions to $h_{\mathcal{A}}, h_{\mathcal{B}}$ and $h_{\mathcal{C}}$, we do so for $z \in \Omega$. To get $C_\varepsilon \to C_0$ technically we must let $z \to \partial\Omega$ *before* the continuum limit has been taken, but this is not so very difficult. First we will rephrase our argument in the (Ω, a, b, c, d) language. Suppose we are interested in a (blue) crossing from the arc $\overline{ab}$ to the arc $\overline{cd}$: this crossing probability can be retrieved as the value of one of the h functions as the distance between its argument and d tends to zero.

Now let Υ be a crosscut in Ω which separates d from c and the boundary arc $\overline{ab}$. Let Ω_0 be the connected component of $\Omega \setminus \Upsilon$ containing d at the boundary. Let $\eta > 0$ and $z \in \Omega_0$ a distance at most η from d and at least $\sqrt{\eta}$ from Υ, then we can set up of the order of $\log \eta$ annuli of fixed modulus, separating z and a point of the prime end d from Υ. The probability of at least one blue circuit (in the relevant portion) of such an annulus differs from unity by some power of η (these sorts of "Russo–Seymour–Welsh" arguments will be described in greater detail later, e.g. in the proof of Lemma 2.6). In the presence of such a circuit, the crossing event given by this "interior" point z would lead to the desired crossing. □

3.3 Limit is Supported on Curves

Here we provide a proof of Lemma 2.5, i.e. any limit point of the μ_ε's is supported on curves. Our proof will utilize three additional lemmas, but first we must discuss crosscuts.

As alluded to several times before, we envision Ω as the conformal image of the upper half plane via some map $\phi : \mathbb{H} \to \Omega$. The prime end a is defined in the usual fashion as the set of all limit points of sequences $\phi(z_n)$, $z_n \to z_a$, where $z_a \in \mathbb{R}$ is fixed. Alternatively, define

$$A_k = \overline{\phi(\{|z - z_a| \leq 1/k, \mathrm{Im} z > 0\})},$$

then the prime end a can be defined as $\cap_k A_k$. We also define

$$\alpha_k = \phi(\{|z - z_a| = 1/k, \mathrm{Im} z > 0\})$$

as the k^{th} crosscut of a. We define similar quantities for b and call them B_k and β_k, respectively. Finally let us also define $\gamma^{k,\varepsilon}$ to be the curve formed by γ^ε from the last exit from A_k to the first entrance into B_k after this last exit from A_k. We remark that for finite k, with non–zero probability, γ^ε *will* form multiple crossings of the region $\Omega_k \equiv \Omega \setminus (A_k \cup B_k)$, but this probability tends to zero as $k \to \infty$, as can be seen by applying Cardy's formula (or by using Russo–Seymour–Welsh type arguments, c.f. the proof of Lemma 2.6).

Lemma 3.8. *Consider the domain Ω_k and let μ_k' be a limit point of the measures on the curves $\gamma^{k,\varepsilon}$. Then the μ_k''s are supported on Hölder continuous curves. Moreover, the weak convergence to μ_k' can be taken with respect to the topology defined by the sup–norm distance between curves.*

Proof. These claims follow from the result of [1]. We claim that on Ω_k, the curves $\{\gamma^{k,\varepsilon}\}$ satisfy hypothesis H1 of [1] namely: The probability of multiple crossings of circular shells (intersected with Ω_k) goes to zero as the multiplicity gets large. This is clear if we consider circular shells with the outer radius sufficiently small, dependent on k. Indeed, for R less than some R_k, there is no possibility of both blue and yellow boundary inside Ω_k intersected with the corresponding circular shell. Thus we must only rule out many crossings of $\gamma^{k,\varepsilon}$ of the circular shell either in the presence of no boundary or in the presence of a monochrome boundary – with the rate of decay which increases to infinity with the number of traversals. These estimates follow from straightforward repeated applications of Lemma 3.4. □

For the next lemma, we need another definition. We say that we have a *jump* of magnitude (at least) ℓ if

$$\gamma^{k+\ell,\varepsilon} \cap (\Omega_\varepsilon \setminus (A_k \cup B_k)) \neq \gamma^\varepsilon \cap (\Omega_\varepsilon \setminus (A_k \cup B_k)).$$

For an illustration see Figure 4.

Lemma 3.9. *For every k the magnitude of the jump stays bounded as $\varepsilon \to 0$ with probability one.*

Proof. The modulus of the conformal rectangle $(A_k \setminus A_{k+\ell}, \alpha_k, \alpha_{k+\ell})$ tends to infinity as $\ell \to \infty$. We observe that in the event of a jump there must be a crossing of this conformal

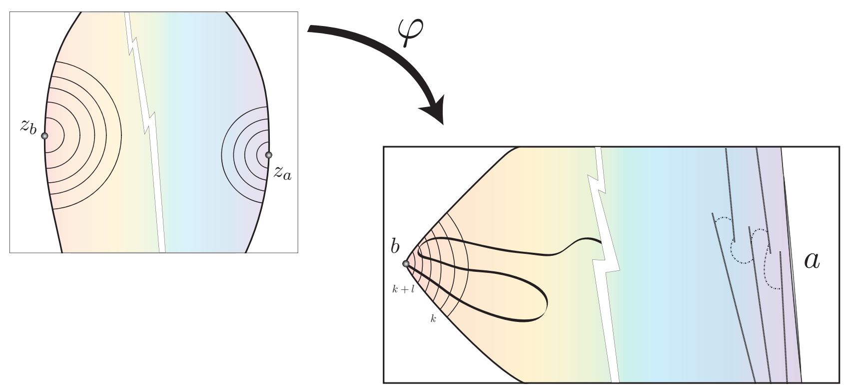

Figure 4: A jump of magnitude l occurring in the vicinity of the prime end b.

rectangle. As $\varepsilon \to 0$, we may utilize Cardy's formula to show that the probability of such a crossing is bounded by some constant $\delta_{k,\ell}$ which tends to zero as $\ell \to \infty$, i.e. as $\varepsilon \to 0$, the probability of jumps of unbounded magnitude is zero. Analogous arguments hold for the B_k's. □

We are now prepared to define the **Dist** function referred to in the statement of Lemma 2.8.

Definition 3.10. Let $\lambda_\ell > 0$ be fixed numbers that satisfy $\sum_\ell \lambda_\ell = 1$, e.g. $\lambda_\ell = 2^{-\ell}$. If γ_r and γ_g are two curves in Ω from a to b, we denote, as before, γ_r^ℓ (or $\gamma_r^{\ell,\varepsilon}$) the portion of the curve in Ω_ℓ, etc. Let $d_\ell(\gamma_r, \gamma_g)$ denote the usual sup norm distances between γ_r^ℓ and γ_g^ℓ. Then we define

$$\mathbf{Dist}(\gamma_r, \gamma_g) = \sum_\ell \lambda_\ell d_\ell(\gamma_r, \gamma_g).$$

Lemma 3.11. *Let $D(\eta, l)$ denote the circular annulus with inner radius η and outer radius l. Consider the event of a (i) 5–arm crossing of $D(\eta, l)$ and (ii) 6–arm crossing of $D(\eta, l)$. Then the 5–arm event has probability bounded above by $(\eta/l)^2$ while the 6–arm event has probability bounded above by $(\eta/l)^{2+\kappa}$ for some $\kappa > 0$.*

Proof. Let us rescale back so that the lattice spacing is of order unity and the diameter of Ω_ε is of order N. Then the five arm event in $D(\eta, l)$ is the event of five crossings between circles of radius ηN and lN. Approximating by square annular regions if necessary, we find that by [7], the probability of this event is bounded above by $\left(\frac{\eta}{l}\right)^2$. (This requires the same proviso about flower disjointness, when necessary, as eluded to in the proof of Lemma 3.6. Such issues were dispensed with in the proof of Lemma 7.3 in [5].) To bound the 6–arm event, we note that if we let A denote the event of one crossing in the annular region, then the probability of A is bounded by $\left(\frac{\eta}{l}\right)^{\kappa}$ by standard Russo–Seymour–Welsh arguments (of the type that will be exemplified in the proof of Lemma 2.6). Then letting B be the event of 5 crossings in the annular region and applying Lemma 3.4 to $A \circ B$ yields the desired result. □

Proof of Lemma 2.5. We first establish that any limiting measure μ' is supported on curves from a to b. By Lemma 3.9, a μ' generic set intersected with $\Omega \setminus (A_k \cup B_k)$ is the same as $\mu'_{k+\ell}$ generic curves (these objects are curves by Lemma 3.8) intersected with $\Omega \setminus (A_k \cup B_k)$ for some ℓ. The family of domains $\Omega \setminus (A_k \cup B_k)$ is monotone and exhaustive. μ' is concentrated on curves by Alexandroff's Theorem. By Lemma 3.9 again, these curves are crosscuts from a to b.

To show that these are Löwner crosscuts it is enough to show that they almost surely satisfy conditions (L1) and (L2). Consider a parametrization of γ with non–vanishing speed. It is not difficult to see that a violation of (L1) implies that there exists some point z_0 which is visited at least three times if z_0 is in the bulk or twice if z_0 is on the boundary. We remind the reader that this is in the continuum; at the lattice level, our collisions could represent approaches which are microscopically large but macroscopically small e.g. a sublinear power of N. Note that each such encounter in the interior leads to 6 arms. If z_0 is a boundary point a more careful counting shows that we either have at least 5 arms or a crossing of a conformal rectangle with arbitrarily large modulus near the boundary, whose probability vanishes. In the case of (L2), we again have a crossing of a conformal rectangle near the boundary with arbitrarily large modulus or a six arm event.

Thus to finish we appeal to Lemma 3.11; let us divide the continuum into a grid of scale η. In order for the described event to occur in the interior of the domain, there must be some l such that the 6–arm event happened in some annulus $D(\eta, l)$. The probability of this is given in Lemma 3.11 and therefore the probability that this happens anywhere is bounded by $(\eta/l)^{2+\kappa}\left(\frac{1}{\eta^2}\right) = \frac{1}{l^2}\left(\frac{\eta}{l}\right)^{\kappa}$. So, ultimately, the probability of such an event is zero.

In order for the event to occur at a boundary grid site, the 5 arm event should occur near the boundary. The number of such sites is bounded by $C\eta^{\delta-2}$ for some $\delta > 0$ and some constant C since the Minkowski dimension of the boundary is strictly less than two. Thus, by Lemma 3.11, the probability of such an event is bounded by η^{δ}, and therefore also vanishes as we remove the grid. □

3.4 Preservation of $M(\partial\Omega) < 2$

Here we show that if we start with some domain Ω with $M(\partial\Omega) < 2$, then the exploration process also yields a curve with Minkowski dimension less than 2.

Proof of Lemma 2.6. Let $z \in \text{Int}(\Omega)$ and $g_\delta(z)$ the box (rhombus) of radius δ surrounding z and $D(z)$ denote the distance between z and $\partial\Omega$. We claim that there is some $\psi > 0$ such that for all ε sufficiently small,

$$\mathbb{P}_\varepsilon(\mathbb{X}_t^\varepsilon \in g_\delta(z)) < C_2 \left(\frac{\delta}{D}\right)^\psi$$

where C_2 is a constant.

This follows from Russo–Seymour–Welsh theory, which we do here in some detail. Indeed, if $r < s$, let $A_{s,r}(z) \equiv B_s(z) \setminus B_r(z)$ denote the rhomboidal annulus centered at z, where, if necessary, the sides are approximated, within ε, by the lattice structure. Assume temporarily that $A_{s,r}(z) \subset \text{Int}(\Omega)$. Clearly, if there is both a yellow and a blue ring in $A_{s,r}$, then $\mathbb{X}_t^\varepsilon$ cannot possibly visit $B_r(z)$ (since the yellow portion of $\mathbb{X}_t^\varepsilon$ cannot penetrate the blue ring and similarly with yellow $\leftrightarrow$ blue). Now by Theorem 3.10, item (iii) in [5], the probability of a blue ring in $A_{M,\lambda M}$ is bounded below uniformly in ε by a strictly positive constant that depends only on λ. Let $\eta > 0$ denote a lower bound on the probability that in $A_{4L,3L}$ there is a blue ring and in $A_{3L,2L}$ a yellow. Now let k satisfy $2^k > \varepsilon^{-1}D > 2^{k-1}$ and similarly $2^\ell > \varepsilon^{-1}\delta > 2^{\ell-1}$. Then, give or take, there are $k - \ell$ independent annuli in which the pair of rings described can occur. The probability that all such ring pair events fail is less than $C_1(1-\eta)^{k-\ell} \leq C_2 \left(\frac{\delta}{D}\right)^\psi$, where C_1 and C_2 are constants and $\psi > 0$ is defined via η.

Let us fix a square grid of scale δ with $\varepsilon << \delta << 1$. Let $\mathcal{N}_\delta$ denote the number of boxes of scale δ that are visited by the process. We claim that for all ε sufficiently small

$$\mathbb{E}_\varepsilon(\mathcal{N}_\delta) \leq C_{\psi'} \left(\frac{1}{\delta}\right)^{2-\psi'} = C_{\psi'} n^{2-\psi'}, \tag{18}$$

where $\psi' > 0$ is a constant and $n = n_\delta = \delta^{-1}$ represents the characteristic scale of Ω on the grid of size δ^{-1}. In particular we may take $\psi' < \min\{\psi, \theta\}$, where $\theta \in [0,1]$ describes the roughness of the boundary: $M(\partial\Omega) = 2 - \theta$.

Let n_k denote the number of boxes a distance $k\delta$ (i.e. k boxes) from $\partial\Omega$ and

$$N_l = \sum_{k \leq l} n_k.$$

Our first claim is that for all δ,

$$N_l < C_{\theta'} n^{2-\theta'} l^{\theta'}, \tag{19}$$

for any $\theta' < \theta$, where $C_{\theta'}$ is a constant. To see this, let us estimate the total area of boxes on a grid of size κ intersected or within one unit of $\partial\Omega$. It is not hard to see that this is bounded by $C_{\theta'} \times \left(\frac{1}{\kappa}\right)^{2-\theta'} \times \kappa^2 = C_{\theta'}\kappa^{\theta'}$, where $C_{\theta'}$ is a constant which is uniform for a fixed

$\theta' < \theta$. Taking $\kappa = l\delta$ and noting that *these* boxes contain all of the $n_1 + \cdots + n_l$ boxes of scale δ (i.e. boxes within l units of $\partial\Omega$), the claim follows.

Now, clearly,

$$\mathbb{E}_\varepsilon(\mathcal{N}_\delta) \leq C_2 \sum_{k=1}^{l_{\max}} n_k \cdot \left(\frac{1}{k}\right)^{\psi}.$$

Let us now dispense with the sum in the display. Summing by parts, we get

$$\sum_{k=1}^{l_{\max}} n_k \left(\frac{1}{k}\right)^{\psi} = N_{l_{\max}} l_{\max}^{-\psi} + \sum_{k=1}^{l_{\max}-1} N_k \left(\frac{1}{k^{\psi}} - \frac{1}{(k+1)^{\psi}}\right).$$

Now if $\psi > \theta$, then $\psi > \theta'$. Using Eq.(19) and pulling out an $n^{2-\theta'}$, the sum is convergent. Meanwhile, the first term (again using the estimate in Eq.(19)) is smaller. Conversely, if $\psi \leq \theta$, then both terms are of order $n^{2-\theta'} l_{\max}^{\theta'-\psi}$ and the result follows if we take $l_{\max} = n$. It is reemphasized that the estimate in Eq.(18) is uniform in ε; by further sacrifice of the constant, we may claim that Eq.(18) holds for all box–scales in the range $[\delta, 2\delta]$.

The remaining argument is now immediate. Letting $\delta_k = 2^{-k}$ we have that for any $\delta \in [\delta_{k+1}, \delta_k]$ and $s > 0$

$$\mathbb{P}_\varepsilon(\mathcal{N}_\delta > C_{\psi'} n_\delta^{2-\psi'+s}) \leq \frac{1}{2^{ks}}. \tag{20}$$

The result follows, for any $s > 0$, by taking $\varepsilon \to 0$ and summing over k. □

3.5 Uniform Continuity

Here we provide a proof of Lemma 2.8. We will need a definition and an auxiliary lemma.

Definition 3.12. Let Ω be a domain. Let $\delta \gg \eta > 0$ and let $\gamma : [0,1] \to \Omega$ be a parametrized curve. We say that γ has a δ–η doubleback if there exists disjoint subsegments I_1 and I_2 of $[0,1]$, with $\text{diam}(\gamma(I_1)) \geq \delta$, $\text{diam}(\gamma(I_2)) \geq \delta$, and such that the segments $\gamma(I_1)$ and $\gamma(I_2)$ are η–close in the sup–norm.

Lemma 3.13 (No Doubleback)**.** *Let Ω denote a domain of the type described, and let $\gamma \in supp(\mu')$. Let δ, η satisfy $\eta < c_1\delta$, with a particular c_1 of order unity. Then for all δ sufficiently small, there are additional constants c_2 and c_3 of order unity such that for all ε sufficiently small, the μ_ε–probability (and hence the μ'–probability) of a δ–η doubleback is bounded above by*

$$\frac{c_2}{\delta^2} \cdot e^{-c_3\delta/\eta}.$$

Proof. To verify the above, it is sufficient to verify the statement in the measures μ_ε for ε sufficiently small. Thus let $\delta \ll 1$ and η small as desired and then ε much smaller than the scale set by η. (We are envisioning that η/δ actually tends to zero.) For k large but of order unity, let us grid the domain Ω into pixels of scale $k^{-1}\delta$. It's not difficult to see that the event in question necessitates an easy–way crossing of a rectangle of this scale with aspect ratio of order unity. If there are two such crossings (of the same curve and therefore not intersecting) a distance – in the sup–norm – η of one another, then by explicit percolation estimates, this probability is of the order $e^{-c_3\delta/\eta}$. Since there are at most the order of δ^{-2} such boxes, the claim follows. □

Proof of Lemma 2.8. We start by treating the problem in a regular domain without the complications associated with prime ends; it may be assumed that the sup norm distance between the curves is simply η; the issues associated with prime ends will be dispensed with at the end of the proof.

We envision Ω to be rectangular, with a as the right bottom corner, and $\eta \ll \delta \ll \Delta$. We will abbreviate $C_\varepsilon(\gamma_1) = C_\varepsilon(\Omega \setminus \gamma_1([0,T]), \gamma_1(T), b, c, d)$ and similarly for $C_\varepsilon(\gamma_2)$. Let us first assume that $C_\varepsilon(\gamma_1) > C_\varepsilon(\gamma_2)$, and let B_δ denote the ball of radius 4δ around $\gamma_1(T)$. Notice that by assumption, B_δ is well away from c. For the sake of argument, let us continue the curve γ_1 in some fashion to the appropriate prime cut around c, so that Ω is divided into two connected components, $\tilde{C}_\ell$ and $\tilde{C}_r$, where $\tilde{C}_\ell$ is the component containing d and $\tilde{C}_r$ is the component containing b. Consider the union of the Hausdorff–η neighborhoods of γ_1 and γ_2, which we denote N_η. We let $\mathcal{C}_\eta = (N_\eta \setminus B_\delta) \cap \tilde{C}_l$, so that the "right" boundary of $\mathcal{C}_\eta$ is contained in γ_1. The "left" boundary of $\mathcal{C}_\eta$, which is $(\partial N_\eta \cap \tilde{C}_\ell) \setminus \partial B_\delta$, we will denote by $\mathcal{L}$.

Now let us consider the event $Y_{\mathcal{L}}$ of a yellow crossing in Ω starting from the arc $[b, c]$ to $\mathcal{L}$ and ending at some point $R \in \mathcal{L}$, not crossing γ_1. There is a path with the property that it ends on the "earliest" such $R \in \mathcal{L}$ (that is, walking along $\mathcal{L}$ starting at a, R is the first point hit by such a yellow crossing) and is the clockwise–most such path. Let us condition on this path and defer to below (in the context of a nearly identical argument involving blue paths) the cases that R is within δ of a boundary. By topological reasons, any blue crossing which contributes to $C_\varepsilon(\gamma_1)$ must pass through a neighborhood of radius η around R, but now we may surround the point R by the order of $\log(\delta/\eta)$ annuli of fixed modulus (there is enough "room" since by Russo–Seymour–Welsh type arguments, R is outside B_δ with high probability), each of which now has an independent probability of order unity of having a yellow circuit (within the unconditioned region). Any one of these circuits is enough to prevent the possibility of a blue crossing for γ_1 (since R is well away from $\gamma_1(T)$). The probability of having at least one such circuit is $1 - \eta^\beta$, for some β. Thus, if B_1 denotes the event of a blue crossing contributing to $C_\varepsilon(\gamma_1)$, then $P(B_1 \mid Y_{\mathcal{L}}) = \eta^{\beta'}$ and so $P(B_1 \cap Y_{\mathcal{L}}^c) \geq C_\varepsilon(\gamma_1) - \eta^{\beta'}$.

Let B_1 denote the event that a blue crossing contributing to $C_\varepsilon(\gamma_1)$ has occurred. Either there is a blue crossing in the complement of γ_1, or any such blue crossing must hit γ_1. We will now show that $C_\varepsilon(\gamma_2 \mid B_1 \cap Y_{\mathcal{L}}^c)$ is of order unity. We claim that if the blue crossing happens in the complement of γ_1, but *not* in the complement of γ_2, then either a blue crossing has also happened for γ_2 or the event $Y_{\mathcal{L}}$ must happen, which would be contrary to our conditioning. Let us therefore focus on the case where the blue crossing hits γ_1. We condition on the clockwise–most such blue crossing which has the property that it ends on the "latest" possible $S \in \gamma_1 \setminus B_\delta$ ("late" again relative to a), and let us denote this crossing Γ_1. By the same Russo–Seymour–Welsh type arguments as before, such a point exists with high probability (if indeed a blue crossing hitting γ_1 exists) and we may also assume in addition that S is a distance of order δ from a. Let $\gamma_1(s) = S$. The crosscut $\Gamma_1 \cup \gamma_1([0,s])$ divides Ω two connected components, C_r and C_ℓ, where C_r contains b and C_ℓ contains d. Also, let us define $\Gamma \equiv \partial C_\ell = \Gamma_1 \cup \gamma_1([0,s])$. By the definition of S, $\gamma_1([s,T]) \subset C_\ell$.

Appealing to Lemma 3.13, let us assume (for δ, η sufficiently small, with μ_ε–probability (for ε small enough) in excess of $1 - \frac{1}{2}\vartheta$) that γ_1 has no δ–(2η) doubleback. Therefore there exists a point in $B_\delta \cap \gamma_1$ which is of distance $> 2\eta$ from $\gamma_1([0,s])$, and hence Γ, since Γ_1 does not enter B_δ. Let $\gamma_1(t^*)$ denote the latest such point (i.e. t^* is maximal). Since γ_2 is η–close to γ_1 in sup–norm, $\gamma_2(t^*) \in C_\ell$ (so in particular γ_2 cannot lie entirely in C_r) *and* $\gamma_2(t^*)$ is of distance $> \eta$ from Γ. Now we notice that by the maximality of t^*, $t^* > s$, since the opposite inequality would necessitate Γ_1 entering B_δ. Since γ_2 lies in a Hausdorff–η neighborhood of γ_1, these facts imply that – since γ_2 cannot cross itself, the "right" side of $\gamma_2([0,s])$ (the side visible from $[a,b]$) is also visible from Γ_1. Therefore, either the blue crossing for γ_2 is automatic, or γ_2 is topologically in a position for the blue crossing from γ_1 to be continued to it.

If there is enough room to set up the order of $\log(\delta/\eta)$ annuli of fixed modulus around the point S (this requires S to be δ away from the boundary) then we may simply use Russo–Seymour–Welsh to conclude that with probability $1 - \eta^\zeta$ for some ζ, Γ_1 can be "continued" to be a blue crossing for γ_2 (this blue crossing cannot intersect γ_2 in C_ℓ – and hence cannot be "broken up" by γ_2 since that would imply a yellow arm from arc $[b,c]$ to $\mathcal{L}$ and hence the occurrence of $Y_{\mathcal{L}}$, contrary to our current conditioning). We will now handle the cases where S happens to be close to some boundary. First let us dispense with the juncture of two boundaries. With high probability the point S is not near the a or b prime end, because with high probability Γ_1 itself will not come close to these places. By fiat, S cannot be near the point c and, as it turns out, if it is near d, then this degenerates into the case where it is near the $[c,d]$ boundary alone. Thus, we may assume that S is near at most one boundary.

Suppose that S occurs near the $[c,d]$ boundary, then by extending Γ_1 to include the proper portion of the $[c,d]$ boundary, it is seen that we may apply a Russo–Seymour–Welsh argument. Next let's discuss the case where S is close to one of the "yellow" boundaries (i.e. $[d,a]$ or $[b,c]$). We define a point $p \in \gamma_1$ to be a choke point if the following happens: Consider the annular region centered on p with inner "radius" δ and outer radius Δ; γ_1 divides the annular region into a "blue" and "yellow" part. Then p is a choke point if the $[d,a]$ or $[b,c]$ boundary intersects the core of the annular region. Obviously, there is a "highest" such choke point on the curve γ_1 and, on this choke point, again by Russo–Seymour–Welsh, there will be a yellow connections between γ_1 and the corresponding boundary, which precludes the possibility of Γ_1. This only fails to occur with probability which is some power of δ/Δ, which is therefore a power of η. Finally, we consider the $[a, b]$ boundary. While for topological reasons this is really not an issue, it is also negligible due to another argument involving choke points.

The upshot is that $C_\varepsilon(\gamma_2 \mid B_1 \cap Y_{\mathcal{L}}^c) \geq 1 - \eta^{\zeta'}$. Since $P(B_1 \cap Y_{\mathcal{L}}^c) \geq C_\varepsilon(\gamma_1) - \eta^{\beta'}$, we conclude that $C_\varepsilon(\gamma_2) \geq C_\varepsilon(\gamma_1) - \eta^{\xi}$, for some $\xi > 0$. We are done in the case that $C_\varepsilon(\gamma_1) > C_\varepsilon(\gamma_2)$. If the opposite inequality holds, then we note that there is either a blue crossing from $[\gamma_1(T), b]$ to $[c, d]$ or a yellow crossing from $[d, \gamma_1(T)]$ to $[b, c]$ (with the same statement for γ_2) and hence the conclusion will follow by exactly the same proof for the yellow crossing problem, with the role of γ_1 and γ_2 interchanged.

To finish the proof, given ϑ, choose L large enough such that the probability of a monochrome connection between Ω_1 and $\Omega \setminus \Omega_L$ is less than say $\vartheta/4$. Foremost we shall run the argument in Ω_L; here, for the analogues of R and S, the objects of concern are now the extreme (highest/lowest) points in Ω_L (this is not a tautology in the event that γ_1 itself leaves Ω_L). If the R, S–type points exist but take place in Ω_L^c, this necessitates a monochrome crossing of the above–mentioned type. Otherwise, there is nothing to worry about to begin with. In any case, the argument in Ω_L yields the bound $C\eta_L^\alpha$, where $\eta_L = d_L(\gamma_1, \gamma_2)$. It is therefore seen, in the worst case scenario, that we are done if we choose η so that $C\left(\frac{\eta}{\lambda_L}\right)^\alpha < \vartheta/4$. □

3.6 *A Priori* Bounds

In this paragraph we prove Lemma 2.9. Let us observe that

Lemma 3.14. *Let $\gamma(t)$ be the chordal SLE generated by $w(t)$. Then*

- *$Im(\gamma(t)) \leq 2\sqrt{t}$.*
- $\sup_{s \leq t} |\gamma_s| \geq \frac{|w_t|}{4}$.

Proof. For the first statement note that $\partial_t(\mathrm{Im}(g_t)) = -2\mathrm{Im}(g_t)/|g_t - w_t|^2 \geq -2/\mathrm{Im}(g_t)$, so $\partial_t(\mathrm{Im}(g_t))^2/4 \geq -1$. Integrating, we get $(\mathrm{Im}(g_t))^2 \geq (\mathrm{Im}(z))^2 - 4t$. So if $\mathrm{Im}(z) > 2\sqrt{t}$ then $\mathrm{Im}(g_t) > 0$.

For the second part, assume that $R = \sup_{s \leq t} |\gamma_s|$. Let $[\mu_t, \lambda_s]$ be the image $g_t(\gamma[0, t])$. Note that $w_t \in [\mu_t, \lambda_t]$ and that $\mu_t \leq 0 \leq \lambda_t$, since μ_t is decreasing and λ_t is increasing and $\mu_0 = \lambda_0 = w_0 = 0$.

Now let $A(z) = z + R^2/z$ be the map of the complement of the half-disk $\{|z| \leq R\}$ to $\mathbb{H}$. Note that A maps the half-circle $\{|z| = R\}$ onto $[-2R, 2R]$. The map $g_t(A^{-1}(z))$ is well-defined, since $\gamma([0, t]) \subset \{|z| \leq R\}$, and maps $[-2R, 2R]$ onto a curve separating $[\mu_t, \lambda_t]$ from infinity. By the monotonicity of harmonic measure, $|w(t)| \leq \lambda_t - \mu_t \leq 4R$, which proves the second statement of the lemma. □

Now we are in a position to prove Lemma 2.9.

Proof of Lemma 2.9. On the basis of the above lemma, $|w_t| > n$ implies that in the half plane a rectangle of aspect ratio of the order $n/\sqrt{t}$ has been crossed by $g_0(\gamma_{[0,t]})$. But this means that $\gamma_{[0,t]}$ itself crossed a *conformal* rectangle with conformal modulus $n/\sqrt{t}$. Invoking Lemma 2.7, the probability of such an event is bounded by $C_1 e^{-C_2 \frac{n}{\sqrt{t}}}$ for some $C_1, C_2 > 0$. □

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