Lecture # 12
Integral Bounds

Now, low point of exclusive reliance on parametric description of line integration.

-- Want to state (and prove) inequality which is obvious from Riemannian construction.

Here, actually requires slight bit of cleverness.

Recall, from 32B there was second type of line integral.

As before, \( \Gamma \) a path and \( g(x,y) \) a function (smooth etc.) on path.
\( \Gamma \) parameterized by \( x = x(t), \ y = y(t) \ ; \ t_1 \leq t \leq t_2. \)
These have derivatives \( \dot{x}(t) \) and \( \dot{y}(t) \) respectively. Then we define

\[
\int_{\Gamma} gds = \int_{t_1}^{t_2} g(x(t), y(t)) \sqrt{\dot{x}^2 + \dot{y}^2} \, dt.
\]

Reminder: This was the type of path integral which did not depend on orientation of path.
Finally, recall that if $g(x,y) \equiv 1$ then $\int_{\Gamma} ds = |\Gamma|$ is called the *arclength* of the curve $\Gamma$.

Now for 132 these second type of line integrals not so important in their own right. But needed to prove other results which *are* important.

Useful claim(s), as far as this course is concerned:

\[ [I] \quad \left| \int_{\Gamma} f(z) \, dz \right| \leq \int_{\Gamma} |f(z)| \, ds \]

\[ [II] \quad \int_{\Gamma} |f(z)| \, ds \leq (|\Gamma|)[F_{\text{max}}(\Gamma)] \]  
where $F_{\text{max}}(\Gamma)$ is the maximum value that $|f(z)|$ takes on along $\Gamma$.

First inequality means: (1) do integral on lhs, will get complex number. Take modulus of that complex number, will get positive real number. That positive real number is smaller than (or equal to) what you would get if you did “other type” of 32B integral where “$g(x,y)$” is $|f(z)|$.

Most often, we use these in tandem -- eliminating the middle term.
Inequality [II] actually trivial -- should be taught in 32B. Do this for general $g(x,y)$ which happens to be positive. Write down line integral of $g$ according to some parameterization:

$$\int_{\Gamma} g(x, y) ds = \int_{t_1}^{t_2} g(x(t), y(t)) \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \leq G \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$$

where $G = \max_{t_1 \leq t \leq t_2} g(x(t), y(t))$ is the maximum value that $g$ achieves on the contour $\Gamma$.

But coefficient of this $G$ is just the length of $\Gamma$ and so we

$$\int_{\Gamma} g(x, y) ds \leq G |\Gamma|$$

just our second inequality for $g = |f(z)|$.

First inequality slightly more serious.
Actually will prove weaker form of [I] with x–tra factor of 2 (or $\sqrt{2}$) out front. More clever derivation gets rid of excess factor. But these numbers not important. will just use these as crude bounds.

Look at real part: $\text{Re}\left[ \int f(z)dz \right] = \int_{t_1}^{t_2} [u(x(t),y(t)) \dot{x} - v(x(t),y(t)) \dot{y}] dt$.

Now, in general, $\left| \int h(q) dq \right| \leq \int \left| h(q) \right| dq$

so $\left| \text{Re}\left[ \int f(z)dz \right] \right| \leq \int_{t_1}^{t_2} \left| [u(x(t),y(t)) \dot{x} - v(x(t),y(t)) \dot{y}] \right| dt$.

So far, everything straight forward. But now, want to borrow from vector theory:
Recall that if $\vec{A}$ and $\vec{B}$ are (2–component) vectors:

$$\vec{A} = (a_1, a_2); \quad \vec{B} = (b_1, b_2)$$

then

$$|\vec{A} \cdot \vec{B}| = |\vec{A}| |\vec{B}| \cos \Theta_{AB} \leq |\vec{A}| |\vec{B}|$$

(where $\Theta_{AB}$ is the angle between $\vec{A}$ and $\vec{B}$).

As far as the numbers $a_1, a_2, b_1$ and $b_2$ are concerned, all of this says:

$$|a_1 b_1 + a_2 b_2| \leq (\sqrt{a_1^2 + a_2^2})(\sqrt{b_1^2 + b_2^2}).$$

So, this is quite general and applies even if the “numbers” $a_k$ and $b_k$ happen to be time dependent, real and imaginary parts of some function, etc.

Thus $|u \dot{x} - v \dot{y}| \leq [u^2 + v^2]^{1/2} [\dot{x}^2 + \dot{y}^2]^{1/2}$ – and this, of course is valid inside the integrand.

So:

$$\int_{t_1}^{t_2} |[u(x(t), y(t))\dot{x} - v(x(t), y(t))\dot{y}]| \, dt \leq \int_{t_1}^{t_2} \sqrt{u^2 + v^2} \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$$

that’s $|f|$ that’s $ds$.

That’s it for real part. Derivation for imaginary part very similar gets us

$$\int_{\Gamma} |f(z)dz| \leq 2\int_{\Gamma} |f(z)| \, ds.$$