

Research Statement

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Introduction

Harmonic analysis is a very general and powerful tool used by number theorists to decompose a complicated function into components: some of which are simple and well-understood, while the others are arithmetically rich and mysterious. In such a decomposition, one hopes to use information about the former to better understand the latter. Such information is obtained by the Fourier inversion and Plancherel formulae.

For example, let \mathbb{H} be the Poincaré upper half plane and Γ be $SL_2(\mathbb{Z})$, which acts on \mathbb{H} . Let P the group of upper triangular matrices in $SL_2(\mathbb{R})$. Atle Selberg established the **spectral decomposition**

$$L^2(\Gamma \backslash \mathbb{H}) = \mathbb{C} \oplus L^2_{cusp}(\Gamma \backslash \mathbb{H}) \oplus L^2_{cont}(\Gamma \backslash \mathbb{H})$$

which decomposes $L^2(\Gamma \backslash \mathbb{H})$ into the residual, cuspidal, and continuous spectra, respectively. The mysterious and arithmetically rich component is the cuspidal spectrum. The key to understanding the continuous and residual spectra is the **Maass-Eisenstein series**

$$E_s(z) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} (\text{Im } \gamma z)^{\frac{1}{2}+s}$$

which converges for $\text{Re } s > \frac{1}{2}$. Selberg meromorphically continued it to \mathbb{C} . The residue of the Maass-Eisenstein series at its only pole spans the residual spectrum. For $s \in i\mathbb{R}$, its continuation is smooth and forms an eigenpacket that exhausts the continuous spectrum. Thus, $L^2_{cont}(\Gamma \backslash \mathbb{H})$ is spanned by “Fourier” transforms called **wave packets**

$$E_\varphi(z) = \frac{1}{2\pi i} \int_{i\mathbb{R}} \varphi(s) E_s(z) ds$$

constructed from smooth, compactly supported functions φ on $i\mathbb{R}$. The **Fourier inversion formula** for wave packets

$$\langle E_\varphi, E_t \rangle_{\Gamma \backslash \mathbb{H}} = \varphi_\#(t)$$

shows one can recover φ from its wave packet E_φ up to a projection $\varphi_\#$. Langlands proved the equivalent **Plancherel formula**:

$$\langle E_\varphi, E_\psi \rangle_{\Gamma \backslash \mathbb{H}} = \frac{1}{2} \langle \varphi_\#, \psi_\# \rangle_{i\mathbb{R}}$$

In effect, one can use the Plancherel formula to identify and remove the contribution of the continuous spectrum in what Selberg called the **trace formula**, a generalization of the Poisson summation formula.

Collette Moeglin and Jean-Loup Waldspurger translated this spectral theory of $\Gamma \backslash \mathbb{H}$ and its connection with Eisenstein series into the more general language of rational reductive adèlic groups.

Research Summary

The Case of $SL_2(\mathbb{R})$. In [9], we give a new proof of the Fourier inversion and Plancherel formulae for the continuous spectrum of $\Gamma \backslash \mathbb{H}$, where Γ is any discrete subgroup of $SL_2(\mathbb{R})$ having one cusp and an orbit space $\Gamma \backslash \mathbb{H}$ with finite area. For example, Γ could be the full modular group $SL_2(\mathbb{Z})$. The new proof abridges Casselman's proof of the Fourier inversion formula. A brief sketch of the new proof is as follows: First, we show the absolute convergence of the integral $\langle E_\varphi, E_t \rangle_{\Gamma \backslash \mathbb{H}}$ for imaginary t . We then write it as a sum of two integrals

$$\langle E_\varphi, E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} = \langle E_\varphi, \Lambda_G^G E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} + \langle E_\varphi, \Lambda_G^P E_{-t} \rangle_{\Gamma \backslash \mathbb{H}}$$

according to the orthogonal decomposition afforded by the truncation operators Λ_G^G and Λ_G^P when the truncation parameter $T > 1$. For $\text{Re } t < -\frac{1}{2}$, the truncated Maass-Eisenstein series $\Lambda_G^G E_{-t}(z)$ can be represented as another Eisenstein series

$$\sum_{\Gamma \cap P \backslash \Gamma} \text{Im}(\gamma z)^{\left(\frac{1}{2}-t\right)} \left(1 - \widetilde{\chi_{(T,\infty)}}(\gamma z)\right) - c(-t) \text{Im}(\gamma z)^{\left(\frac{1}{2}+t\right)} \widetilde{\chi_{(T,\infty)}}(\gamma z)$$

where $\chi_{(T,\infty)}$ is the characteristic function of (T, ∞) , $T > 1$, $\widetilde{\chi_{(T,\infty)}}(z) = \chi_{(T,\infty)}(\text{Im } z)$, and $c(t)$ is the Harish-Chandra c -function. When $t \in i\mathbb{R}$, $c(t)$ satisfies

$$|c(t)|^2 = c(t)c(-t) = 1$$

The Eisenstein series satisfies the functional equation

$$E_t = c(t)E_{-t}$$

In the case where $\Gamma = SL_2(\mathbb{Z})$, the Harish-Chandra c -function is given explicitly by

$$c(t) = \frac{\xi(2t)}{\xi(1+2t)}$$

where $\xi(t) = \pi^{-\frac{t}{2}} \Gamma\left(\frac{t}{2}\right) \zeta(t)$ is the completed Riemann zeta-function. We then use the adjoint relation to express the integral $\langle E_\varphi, \Lambda_G^G E_{-t} \rangle_{\Gamma \backslash \mathbb{H}}$ as a linear combination of partial Mellin transforms of the “shifted

by $y^{\frac{1}{2}}$ inverse Mellin transform $\check{\varphi}_\#$ of the function $\varphi_\#(t) = \varphi(t) + c(-t)\varphi(-t)$:

(1)

$$\langle E_\varphi, \Lambda_G^G E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} = \int_0^T y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^{-t} \frac{dy}{y} - c(-t) \int_T^\infty y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^t \frac{dy}{y}$$

The left-hand side is known to be meromorphic in t and holomorphic for $\operatorname{Re} t = 0$. The following integrals

$$\int_0^T y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^{-t} \frac{dy}{y}$$

$$\int_T^\infty y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^t \frac{dy}{y}$$

are holomorphic for $\operatorname{Re} t < -\frac{1}{2}$. Since $y^{-\frac{1}{2}} \check{\varphi}_\#(y)$ is a Schwartz function on \mathbb{R} composed with $\log y$, these integrals are holomorphic for $\operatorname{Re} t < 0$ and continuous for $\operatorname{Re} t \leq 0$. The right-hand side of (1) is meromorphic in t for $\operatorname{Re} t < 0$ and continuous for $\operatorname{Re} t = 0$. Therefore, since both sides of (1) are defined with t varying continuously for $\operatorname{Re} t \leq 0$ (excluding possible poles in the region $\operatorname{Re} t < 0$), the formula

(2)

$$\langle E_\varphi, \Lambda_G^G E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} = \int_0^T y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^{-t} \frac{dy}{y} - c(-t) \int_T^\infty y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^t \frac{dy}{y}$$

also holds for $t \in i\mathbb{R}$! By the adjoint relation, we get the formula

(3)

$$\langle E_\varphi, \Lambda_G^P E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} = \int_T^\infty y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^{-t} \frac{dy}{y} + c(-t) \int_0^T y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^t \frac{dy}{y}$$

for $t \in i\mathbb{R}$ since

$$(4) \quad \Lambda_G^P E_{-t} = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \chi_{(T, \infty)}(E_{-t})_0(\operatorname{Im} \gamma z)$$

has a finite number of nonzero summands. Therefore, for $t \in i\mathbb{R}$

$$\begin{aligned} \langle E_\varphi, E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} &= \langle E_\varphi, \Lambda_G^G E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} + \langle E_\varphi, \Lambda_G^P E_{-t} \rangle_{\Gamma \backslash \mathbb{H}} \\ &= \int_0^\infty y^{-\frac{1}{2}} \check{\varphi}_\#(y) y^{-t} \frac{dy}{y} \\ &= \varphi_\#(t) \end{aligned}$$

This Fourier inversion formula leads to the Plancherel formula

$$\langle E_\varphi, E_\psi \rangle_{\Gamma \backslash \mathbb{H}} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi(t) \langle E_t, E_\psi \rangle_{\Gamma \backslash \mathbb{H}} dt = \langle \varphi, \psi_\# \rangle_{i\mathbb{R}} = \frac{1}{2} \langle \varphi_\#, \psi_\# \rangle_{i\mathbb{R}}$$

The case of rational reductive adèlic groups. In this section, we follow the notation in [1]. In [10], we generalize the proof in [9] for cuspidal Eisenstein series defined on an arbitrary rational reductive adèlic group, giving us the following Fourier inversion formula:

Theorem 1. *Assume π and π' are cuspidal. Suppose $P, P' \in \mathcal{P}_\chi$, $\varphi_P(\lambda) \in \mathcal{H}_P^0(\pi)_\chi$, and $\psi' \in \mathcal{H}_{P'}^0(\pi')_\chi$. Let $\mu' \in \mathfrak{ia}_{P'}^*$. Define $\varphi_{\#}^{P,P'}(\mu')$ to be the function in $\mathcal{H}_{P'}^0(\pi')_\chi$ given by*

$$\varphi_{\#}^{P,P'}(\mu', g) = \sum_{w \in \Omega(P, P')} M(w, w^{-1}\mu') \varphi_P(w^{-1}\mu', m^1k)$$

where $g = nam^1k$ is written according the Langlands decomposition:

$$G(\mathbb{A})^1 = N_{P'} A_{P'} M_{P'}^1 K$$

Then for the wave packet E_φ defined on $G(F) \backslash G(\mathbb{A})$ by

$$E_\varphi(g) = \int_{\mathfrak{ia}_P^*} E(P, \varphi_P(\lambda), \lambda)(g) d\lambda$$

and the meromorphic continuation $E(P', \psi', \mu')$ of the following Eisenstein series

$$E^G(P', \psi', \lambda')(g) = \sum_{P'(F) \backslash G(F)} \psi'(\gamma g) e^{\langle \rho_{P'} + \lambda', H_{P'}(\gamma g) \rangle},$$

we have the Fourier inversion formula:

$$\langle E_\varphi(P), E(P', \psi', \mu') \rangle_{G(F) \backslash G(\mathbb{A})^1} = \langle \varphi_{\#}^{P,P'}(\mu'), \psi' \rangle_{M_{P'}(F) \backslash M_{P'}(\mathbb{A})^1 K}$$

Corollary 2. *(Plancherel Formula)*

Assume π and π' are cuspidal. Suppose $P, P' \in \mathcal{P}_\chi$, $\varphi_P(\lambda) \in \mathcal{H}_P^0(\pi)_\chi$, and $\psi_{P'}(\lambda') \in \mathcal{H}_{P'}^0(\pi')_\chi$. Then for the wave packets E_φ and E_ψ , we have:

$$\langle E_\varphi, E_\psi \rangle_{G(F) \backslash G(\mathbb{A})^1} = \sum_Q n(A_Q)^{-1} \int_{\mathfrak{ia}_Q^*} \langle \varphi_{\#}^{P,Q}(\mu), \psi_{\#}^{P',Q}(\mu) \rangle_{M_Q(F) \backslash M_Q(\mathbb{A})^1 K} d\mu$$

where $n(A_Q)$ is the number of chambers in the Lie algebra \mathfrak{a}_Q of the torus A_Q .

Future Plans

For general rational reductive adèlic groups, we are currently adapting this technique for Eisenstein series built from functions that may not be purely cuspidal. The main difficulty in this setting is overcoming the complexity of the constant term of an Eisenstein series that may not be

cuspidal. However, by using an idea provided by Bill Casselman which involves defining a truncation operator that only truncates the non- L^2 components of the non-cuspidal Eisenstein series, we hope to show the L^2 components present in the formula yield integrals that cancel, thus simplifying the Fourier inversion formula. For reductive p -adic groups, we are working on generalizing the proof for wave packets built from more general Eisenstein integrals.

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