

# Statement of Research Interests

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### Introduction

Atle Selberg showed that the classical space  $L^2(\Gamma \backslash \mathbb{H})$  decomposes according to the hyperbolic Laplacian whose spectrum of eigenvalues consists of a discrete set and a continuum:

$$L^2(\Gamma \backslash \mathbb{H}) = L^2_{disc}(\Gamma \backslash \mathbb{H}) \oplus L^2_{cont}(\Gamma \backslash \mathbb{H})$$

Let  $P$  the group of upper triangular matrices in  $SL_2(\mathbb{R})$ . If  $z \in \mathbb{H}$ , the **Maass-Eisenstein series**

$$E_s(z) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} (\text{Im } \gamma z)^{\frac{1}{2}+s}$$

converges for  $\text{Re } s > \frac{1}{2}$  and has meromorphic continuation to  $\mathbb{C}$ . For  $s \in i\mathbb{R}$ , it is smooth and forms an eigenpacket that exhausts the continuous spectrum. Thus,  $L^2_{cont}(\Gamma \backslash \mathbb{H})$  is spanned by **wave packets**

$$E_\varphi(z) = \int_{i\mathbb{R}} \varphi(s) E_s(z) ds$$

where  $\varphi$  is smooth, compactly supported function on  $i\mathbb{R}$ . Langlands proved the **Plancherel formula** for wave packets:

$$\langle E_\varphi, E_\psi \rangle_{\Gamma \backslash \mathbb{H}} = \langle \varphi, \psi_\# \rangle_{i\mathbb{R}}$$

where  $\psi_\#$  is a projection of a smooth, compactly supported function  $\psi$  on  $i\mathbb{R}$ . It is equivalent to Casselman's **Fourier inversion formula**

$$\langle E_\varphi, E_t \rangle_{\Gamma \backslash \mathbb{H}} = \varphi_\#(t)$$

which states that one can recover the projection  $\varphi_\#$  of a smooth compactly-supported function  $\varphi$  defined on the imaginary axis by integrating its wave packet  $E_\varphi$  against the Maass-Eisenstein series  $E_t$ .

The residue of the Maass-Eisenstein series at its only pole spans the space of constant functions. Thus

$$L^2(\Gamma \backslash \mathbb{H}) = \mathbb{C} \oplus L^2_{cusp}(\Gamma \backslash \mathbb{H}) \oplus L^2_{cont}(\Gamma \backslash \mathbb{H})$$

where  $L^2_{cusp}(\Gamma \backslash \mathbb{H})$  is the subspace of  $L^2_{disc}(\Gamma \backslash \mathbb{H})$  given by the cuspidal spectrum. Eisenstein series, which are built from functions in the discrete spectrum, are the key to understanding the spectral decomposition of automorphic forms and the arithmetic secrets hidden inside the cuspidal spectrum. Mœglin and Waldspurger translated this spectral theory of automorphic forms and Langlands' Plancherel formula into the language of adèlic rational reductive groups.

## Research Summary

We now summarize our work to date. In [8], we give a new proof of the Fourier inversion formula and generalize it for cuspidal Eisenstein series defined on an arbitrary rational reductive adèlic group. The new proof abridges Casselman's proof of the Fourier inversion formula. In [7], we developed an analog of the proof for spherical unramified principal series of reductive  $p$ -adic groups.

### The Proof in the Case of $\Gamma \backslash \mathbb{H}$

Let  $G = SL_2(\mathbb{R})$ ,  $\Gamma = SL_2(\mathbb{Z})$ , and  $K$  denote the maximal compact subgroup  $SO_2(\mathbb{R})$ . There are precisely two **standard parabolic** subgroups, namely the **minimal** parabolic  $P$  of upper-triangular matrices and  $G$  itself.  $P$  decomposes into two subgroups  $NA$ , where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Every  $g \in G$  can be uniquely expressed as  $nak$ , where  $n \in N$ ,  $a \in A$ , and  $k \in K$ . If

$$a = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A$$

let  $H(nak)$  denote the element

$$\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}$$

where  $\mathfrak{a}$  is the Lie algebra of  $A$ :

$$\mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Since  $\mathfrak{a}$  is a one-dimensional real vector space, we may consider its dual space  $\mathfrak{a}^* \cong \mathbb{R}$  and its complexified dual space  $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ . There is one simple root  $\alpha$  and it spans  $\mathfrak{a}^*$ .

$$\alpha(a) = t$$

Letting  $y = e^t$ , we see that  $G/K \cong \mathbb{H}$ . Since the Maass-Eisenstein series is left  $N$ -invariant, it is periodic in  $x$  and therefore has a Fourier series expansion. The **constant term** is the function

$$(E_s)_P(g) = e^{(\frac{1}{2}+s)(H(a))} + c(s)e^{(\frac{1}{2}-s)(H(a))}$$

where  $c(s)$  is the **Harish-Chandra  $c$  function**. For  $s \in i\mathbb{R}$ , we know  $c(s)$  is smooth and satisfies  $c(s)c(-s) = 1$ . The periodicity of the

Eisenstein series implies periodicity of the wave packet. The constant term of the wave packet is

$$(E_\varphi)_P(g) = \int_{i\mathbb{R}} \varphi_\#(s) e^{(\frac{1}{2}+s)(H(a))} ds$$

which is the inverse Mellin transform  $\check{\varphi}_\#$  of  $\varphi_\#$ . The projection  $\varphi_\#(s) = \varphi(s) + c(-s)\varphi(-s)$  satisfies  $\varphi_\#(-s) = c(s)\varphi_\#(s)$ .

Let  $T > 0$  so  $T \in \mathfrak{a}^+$ , the **positive Weyl chamber**. Let  $\text{Re } s > \frac{1}{2}$ . The **truncation**  $\Lambda^T E_s(g)$  is the Eisenstein series

$$\sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{(\frac{1}{2}+s)(H(\gamma g))} (1 - \tau(H(\gamma g) - T)) - c(s) e^{(\frac{1}{2}-s)(H(\gamma g))} \tau(H(\gamma g) - T)$$

Let  $C^T E_s(g)$  be the Eisenstein series

$$\sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} \left[ e^{(\frac{1}{2}+s)(H(\gamma g))} + c(s) e^{(\frac{1}{2}-s)(H(\gamma g))} \right] \tau(H(\gamma g) - T)$$

where  $\tau$  is the characteristic function of the positive reals.

The (non-Hermitian) scalar product  $\langle E_\varphi, E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})}$  converges when  $s \in i\mathbb{R}$  while the Maass-Eisenstein series  $E_{-s}$  converges for  $\text{Re } s < -\frac{1}{2}$ .

Suppose first that  $\text{Re } s < -\frac{1}{2}$ . Then  $\langle E_\varphi, \Lambda^T E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})}$  converges and equals

$$\int_{N(\mathbb{R}) \backslash G(\mathbb{R})} (E_\varphi)_P(g) [e^{(\frac{1}{2}-s)(H(g))} (1 - \tau(H(g) - T)) - c(-s) e^{(\frac{1}{2}+s)(H(g))} \tau(H(g) - T)] dg$$

by the **adjoint relation**. Here,  $dg = e^{-(H(a))} dadk$ . We claim this equality also holds for  $s \in i\mathbb{R}$ :

Recall  $(E_\varphi)_P(g) = \check{\varphi}_\#(g) = \check{\varphi}_\#(a)$  and  $H(g) = H(a)$ . The integrals

$$\int_A \check{\varphi}_\#(a) e^{(-\frac{1}{2}-s)(H(a))} (1 - \tau(H(a) - T)) da$$

$$\int_A \check{\varphi}_\#(a) e^{(-\frac{1}{2}+s)(H(a))} \tau(H(a) - T) da$$

converge uniformly for  $\text{Re } s < -\frac{1}{2}$  and extend continuously to  $\text{Re } s \leq 0$  since  $\check{\varphi}_\#(a) e^{-\frac{1}{2}(H(a))}$  is Schwartz on  $A$ . On the other hand,  $\langle E_\varphi, \Lambda^T E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})}$  is continuous in  $s$  for  $\text{Re } s \leq 0$  also. Thus both

sides are continuous in  $s$  for  $\operatorname{Re} s \leq 0$  and agree in that domain. Thus the equality holds for  $s \in i\mathbb{R}$ . Similarly,  $\langle E_\varphi, C^T E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})}$  equals

$$\int_A \check{\varphi}_\#(a) \left[ e^{(-\frac{1}{2}-s)(H(a))} + c(-s)e^{(-\frac{1}{2}+s)(H(a))} \right] \tau(H(a) - T) da$$

for  $s \in i\mathbb{R}$ . Since  $E_{-s} = \Lambda^T E_{-s} + C^T E_{-s}$ , we combine integrals and by classical Mellin inversion,

$$\langle E_\varphi, E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})} = \int_A \check{\varphi}_\#(g) e^{(-\frac{1}{2}-s)(H(a))} da = \hat{\varphi}_\#(s) = \varphi_\#(s)$$

which is the Fourier inversion formula for wave packets since the Hermitian Petersson inner product  $\langle E_\varphi, E_s \rangle_{\Gamma \backslash \mathbb{H}}$  equals the non-Hermitian inner product  $\langle E_\varphi, E_{-s} \rangle_{G(\mathbb{Z}) \backslash G(\mathbb{R})}$ .

## Future Plans

For general rational reductive adèlic groups, we are currently adapting this technique for Eisenstein series built from functions that may not be purely cuspidal. For reductive  $p$ -adic groups, we are working on generalizing the proof for wave packets built from more general Eisenstein integrals. We are also in the process of writing an expository article on admissible representations of  $p$ -adic groups using the notes based on a two-quarter graduate course we gave in 2007.

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