Date	Speaker/Topic
Jan. 19	Rowan Killip *Overview.*
Jan. 26	Norbet Požár *Local well-posedness in 2D with surface tension.*
Feb. 2	Paul Smith *Local smoothing.*
Feb. 9	Zaher Hani *Local well-posedness in 2D without surface tension I.*
Feb. 16	Zaher Hani *Local well-posedness in 2D without surface tension II.*
Feb. 23	Helen Lei *The Dirichlet to Neumann map.*
Mar. 1	Yao Yao *Taylor instability and the linearized problem.*

Equations of an incompressible fluid.

- Incompressible $=$ ρ is independent of p (and T ←temperature).

- Is water incompressible?

Essentially: $\kappa_T := -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = 4.6 \times 10^{-10}\,\mathrm{N}^{-1}\mathrm{m}^2$.

Thus, a 1 part in 10^3 change in density requires

$$\begin{aligned} & 2.1 \times 10^6\ \mathrm{Nm}^{-2} \\ \equiv\ & \text{a column of water 220m high} \\ \equiv\ & \text{230kg } (\sim 500\,\text{lbs}) \text{ atop a wine cork } (\varnothing = 18\text{mm}) \end{aligned}$$

Notes:
1. $\kappa_S = \kappa_T C_V / C_P = 0.993 \times \kappa_T$ for water. ←Adiabatic compressibility
2. $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = (1482.3\,\mathrm{ms}^{-1})^2$ for water. ←Sound speed
3. All data is for water at 20°C (=68F).

Equations of an incompressible fluid.

(Eulerian formulation)

Newton says: $F = ma$ (for a particle!)

$$\rho\Big[\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u}\Big] = \rho D_t \vec{u} = -g\rho\vec{e}_3 - \nabla p + \eta\Delta\vec{u} \qquad (1)$$

$\vec{u} =$ velocity

$\rho =$ density

$g =$ acceleration due to gravity

$p =$ pressure

$\eta =$ dynamic viscosity ($1.0 \times 10^{-4}\,\mathrm{Nm}^{-2}\mathrm{s}$ for water 20°C)

Conservation of matter: $\frac{d\rho}{dt} + \nabla\cdot(\rho\vec{u}) = 0.$ (2)

Incompressibility means: $\rho =$ const. (3)

Equations of an incompressible fluid.

Combining these gives the (incompressible) Navier–Stokes system:

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -g\vec{e}_3 - \nabla\wp + \nu\Delta\vec{u} \tag{4}$$

$$\nabla \cdot \vec{u} = 0 \tag{5}$$

where:
$\nu = \eta/\rho =$ kinematic viscosity; and
$\wp = p/\rho$ is pressure/density.

(OED: Advection = transfer of material, heat, etc., brought about by . . . mass movement.)

Boundary conditions:

Fluid cannot enter a rigid boundary: $\vec{n} \cdot \vec{u} = 0$

Viscosity inhibits slippage at rigid boundary: $\vec{u} = 0$

The free boundary follows the fluid: *tautology*

Atmosphere *above* the free boundary: $p = p_0$

Viscosity prevents shear at the free boundary:

$$(\vec{n} \cdot \nabla)\Big[\vec{u} - (\vec{n} \cdot \vec{u})\vec{n}\Big] = 0.$$

Aside: what is (Newtonian) viscosity?

Friction: atomic-level phenomena dissipate energy in proportional to velocity difference squared.

Viscosity: energy dissipated in proportion to the square of the *irrotational* velocity gradient:

$$\frac{d}{dt}\int \tfrac{1}{2}\rho|\vec{u}|^2 - \int \rho g z = -\tfrac{\eta}{2}\int (u_{k,j} + u_{j,k})(u_{k,j} + u_{j,k}) \qquad (6)$$

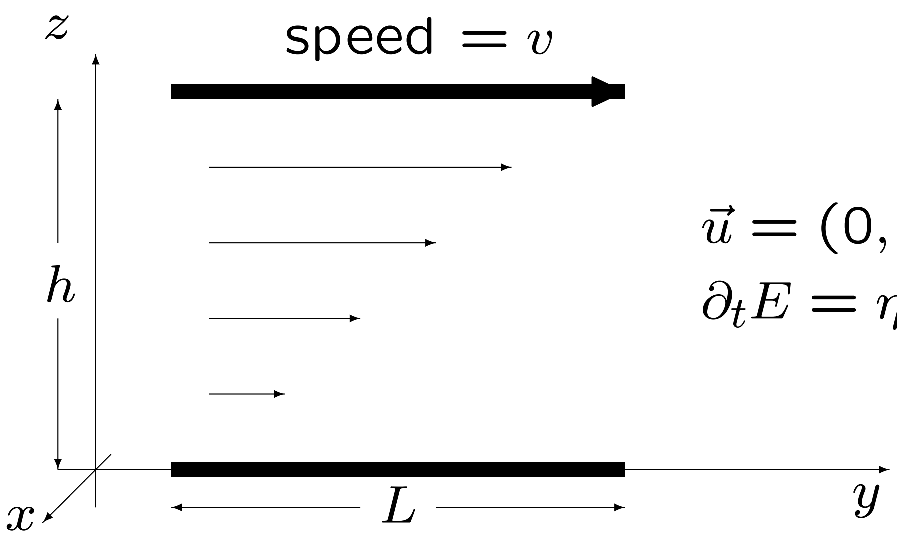
$$= \eta \int \vec{u} \cdot \Delta \vec{u} \qquad (7)$$

(ignoring boundary terms and using $\nabla \cdot u = 0$).

(Note: pre-comma subscripts = components
post-comma = derivatives
repetition = summation)

Aside: measuring viscosity

A fluid between sliding plates



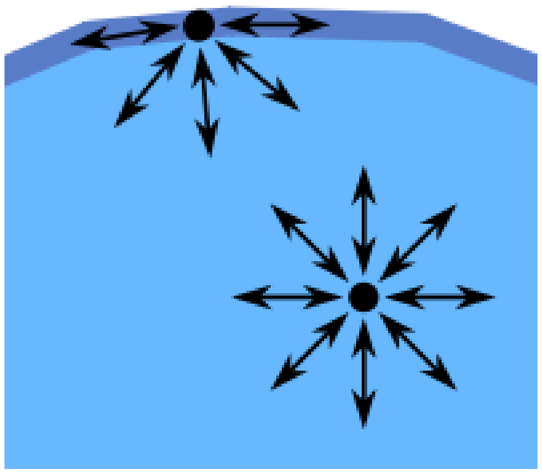
$$\vec{u} = (0, vz/h, 0)$$
$$\partial_t E = \eta v^2 L^2 / h \qquad \text{in an } L \times L \times h \text{ box}$$

$\therefore\ \eta =$ power required (per area of plate) to maintain unit speed difference across a film of unit width

$= 1.0 \times 10^{-4}\,\mathrm{Nm}^{-2}\mathrm{s}$ for water 20°C

Surface Tension.

There is an energy penalty proportional to the water surface area resulting from missing inter-molecular bonds.



(Image from Wikipedia)

Correspondingly, there is a pressure proportional to the mean curvature (the first variation of area) at the surface in the direction of the center of curvature.

For a graph, $z = h(x, y)$, we have

$$2H = -\nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \tag{8}$$

$$= -\Delta h \qquad \text{when } \nabla h = 0 \tag{9}$$

(positive at a crest; negative in a trough).

Conclusion: $p = p_0 + 2\gamma H$ immediately below surface.

$\gamma = 7.27 \times 10^{-2}\,\mathrm{Nm}^{-1}$ for water at 20°C
$\sigma = \gamma/\rho = 7.28 \times 10^{-5}\,\mathrm{m}^3\mathrm{s}^{-2}$ so $\wp = \wp_0 + 2\sigma H$.

Water waves equations without viscosity

Incompressible Euler inside the fluid region:

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -g\vec{e}_3 - \nabla\wp \tag{10}$$

$$\nabla \cdot \vec{u} = 0 \tag{11}$$

At the sea floor:

$$\vec{n} \cdot \vec{u} = 0 \tag{12}$$

On the free surface $\Gamma(t)$:

$$\wp = 2\sigma H(\Gamma) \quad \text{and} \quad \tfrac{d}{dt}\Gamma = (\vec{n} \cdot \vec{u})\vec{n} \tag{13}$$

Note on pressure and vorticity.

• Taking the divergence of the Euler Equation (10) and using incompressibility yields

$$\Delta \wp = -\nabla \cdot \Big[(\vec{u} \cdot \nabla) \vec{u} \Big] = -u_{j,k} u_{k,j} \tag{14}$$

Thus the pressure is determined by an elliptic equation. This represents the infinitude of sound speed.

• Taking the curl yields the vorticity equation

$$\partial_t \vec{w} + (u \cdot \nabla) \vec{w} = (w \cdot \nabla) u \tag{15}$$

where $\vec{w} = \nabla \times \vec{u}$.
Note: $\vec{w}(0) = 0 \;\Rightarrow\; \vec{w}(t) = 0$.

Irrotational fluid motion.

If $\nabla \times \vec{u} = 0$ initially, the flow remains irrotational.

By vector calculus we are guaranteed the existence of a velocity potential $\phi(t, x, y, z)$ such that $\vec{u} = \nabla\phi$.

Incompressibility, $\nabla \cdot \vec{u} = 0$, then implies

$$\Delta\phi = 0 \tag{16}$$

that is, ϕ is *harmonic!*

In particular, the interior motion of the fluid is entirely determined by its behaviour at the boundaries.

Warning: Even for $\vec{u}|_\Gamma \in C_c^\infty$, $\phi|_\Gamma$ may not decay.

Irrotational water waves.

Inside the fluid region: $$\Delta\phi = 0. \tag{17}$$

At the sea floor: $$\vec{n}\cdot\nabla\phi = 0. \tag{18}$$

On the free surface $\Gamma(t)$:

$$\frac{d\phi}{dt} + \tfrac{1}{2}|\nabla\phi|^2 = -gz - 2\sigma H(\Gamma) \tag{19}$$

and

$$\tfrac{d}{dt}\Gamma = (\vec{n}\cdot\nabla\phi)\vec{n} \tag{20}$$

Note: we can reduce to just two unknowns: Γ and $\phi|_\Gamma$.

Dirichlet to Neumann map

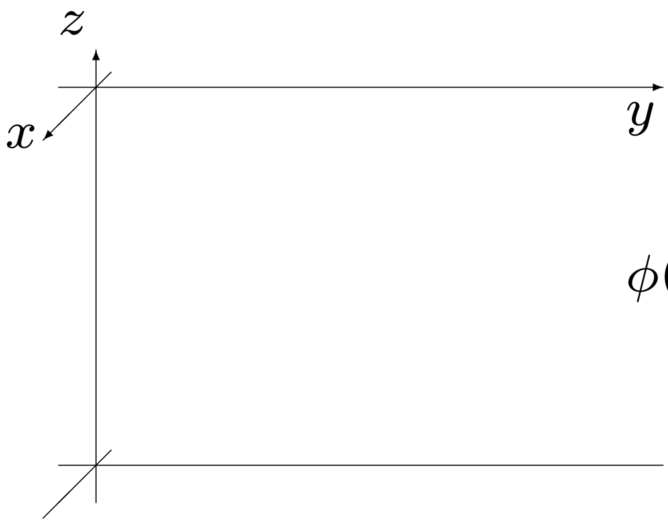
Given Γ and $\phi|_\Gamma$, we need to recover $(\nabla\phi)|_\Gamma$, or rather the only missing piece $\vec{n}\cdot\nabla\phi$.

For elliptic equations, the boundary values are called Dirichlet data; the normal derivatives, Neumann data.

Naturally, the mapping depends intrinsically on the geometry of the region, as dictated by the shape of the sea floor and of Γ.

As a warm up lets consider flat water: $-d \le z \le 0$:

Dirichlet to Neumann map for flat water over a level bottom.



$$\phi(x,y,0) = \psi(x,y) \text{ known}$$

$$\phi(x,y,z) = \iint \frac{\cosh[|\xi|(z+d)]}{\cosh[|\xi|d]} e^{i\xi\cdot(x,y)}\, \widehat{\psi}(\xi)\, d\xi$$

$$\partial_z \phi = 0 \text{ at } z = -d$$

Thus

$$\partial_z \phi(x,y,0) = |\nabla| \tanh(|\nabla| d)\phi(x,y,0) \tag{21}$$

$$E_{\text{kin}} = \langle \psi,\ |\nabla| \tanh(|\nabla| d)\psi \rangle_{L^2} \tag{22}$$

Linearization around still water

Let Γ be $z = h(t; x, y)$ with h small and suppose $\nabla\phi$ is also small.

Then setting $\psi = \phi|_\Gamma$ in

$$\frac{d\phi}{dt} + \tfrac{1}{2}|\nabla\phi|^2 = -gz - 2\sigma H(\Gamma) \quad \& \quad \tfrac{d}{dt}\Gamma = (\vec{n} \cdot \nabla\phi)\vec{n} \quad (23)$$

leads (in the above approximation) to

$$\frac{d\psi}{dt} = -gh + \sigma\Delta h \quad \& \quad \tfrac{d}{dt}h = \partial_z\phi \quad (24)$$

Combining this with $\partial_z\phi = |\nabla| \tanh(|\nabla|d)\psi$, we obtain

$$\partial_t^2\psi = \left(g|\nabla| + \sigma|\nabla|^3\right) \tanh(|\nabla|d)\psi$$

Linearized waves on still water

Substituting the ansatz

$$\psi = \cos(\omega t + kx)$$

reveals the dispersion relation:

$$\omega^2 = (gk + \sigma k^3)\,\tanh(kd)$$

The transition from gravity waves to capillary waves occurs for wavelengths $\lambda \sim 2\pi\sqrt{\sigma/g} \sim 17\,\text{mm}$ (water).

More on gravity waves.

$$\omega^2 = gk\,\tanh(kd)$$

Group velocity $\frac{d\omega}{dk}$ is decreasing in depth

reaching a maximum of $\frac{d\omega}{dk} = \sqrt{\frac{g}{k}}$ when $d = \infty$.

By comparison:

A tsunami with $\lambda \sim 100\,\text{km}$ in deep water (4000m)

travels at $\frac{d\omega}{dk} \sim \sqrt{gd} \sim 200\,\text{ms}^{-1} \sim 700\,\text{km/h}$

More on capillary waves (deepish water).

$$\omega^2 = \sigma k^3$$

Group velocity $\frac{d\omega}{dk} \propto \sqrt{k}$.

Fast waves spend little time near the origin:

$$\iint \langle x \rangle^{-1-\varepsilon} \, |\vec{u}|^2 \, dx \, dt \lesssim E/\sqrt{k}$$

with $E =$ Energy.

$\rightsquigarrow$ Expect $\frac{1}{4}$-derivative local smoothing.

The Rayleigh-Taylor Instability.

In the movies:

Surface Tension in Fluid Mechanics at 15:40.

Flow Instabilities at 17:35.

Or in print:

G. Taylor The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I.

↑Theory

D. J. Lewis The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. II.

↑Experiment

Other formulations ($d = \infty$).

1. Hamiltonian framework; cf. V. Zakharov *Stability of periodic waves of finite amplitude on the surface of a deep fluid.*

The energy is given by

$$E = \tfrac{1}{2}\iiint_{z\le h} |\nabla\phi|^2 + \iint \frac{1}{2} g h^2 + \iint 2\sigma\Big[\sqrt{1+|\nabla h|^2} - 1\Big]$$

or via Green's Theorem,

$$= \tfrac{1}{2}\iint \psi(\vec{n}\cdot\nabla\phi)\sqrt{1+|\nabla h|^2} + \iint \frac{1}{2} g h^2 + \iint 2\sigma\Big[\sqrt{1+|\nabla h|^2} - 1\Big]$$

and the equations are

$$\frac{\partial h}{\partial t} = \frac{\delta E}{\delta \psi} \quad \& \quad \frac{\partial \psi}{\partial t} = -\frac{\delta E}{\delta h} \tag{25}$$

Other formulations ($d = \infty$).

2. Lagrangian coordinate formulation ($2D$, $\sigma = 0$).

Graph $\Gamma(t)$ by $\qquad \vec{x}(t,\alpha) := \Big(x(t,\alpha),\ z(t,\alpha)\Big)$
denoting the position of surface particles (indexed by α).

Newton: $\vec{x}_{tt} = -ge_3 - \nabla\wp$.
Atmosphere: $\wp = 0$ on surface $\Rightarrow \vec{x}_\alpha \cdot \nabla\wp = 0$.
Velocity potential: $\vec{x}_t = \nabla\phi \Rightarrow z_t = Kx_t$ where K is
the (rotated) tangential to normal derivative map
for Laplace's eqn in the geometry dictated by Γ.
In the case of flat water (and $d = \infty$), (21) gives

$$\partial_z\phi = |\partial_x|\phi = \tfrac{1}{\pi x} * \partial_x\phi$$

This convolution operator is the *Hilbert transform*.

Lagrangian coordinate formulation (cont.).

$$x_\alpha x_{tt} + z_\alpha(1 + z_{tt}) = 0 \quad \& \quad z_t = K x_t \tag{26}$$

Attempting to solve this system leads to the requirement

$$\vec{n} \cdot (\vec{x}_{tt} + g e_3) > 0 \tag{27}$$

which expresses the Rayleigh–Taylor stability criterion.

Understanding the appearance of this condition and its role in the analysis is an important goal for this quarter.