32B Killip — Homework 7 — Due May 29

(1) From Section 17.5: 2, 6, 14, 16, and 29.

(2) Let f and g be scalar functions defined throughout the plane. Show that

$$\oint_C g\nabla f \, d\mathbf{r} = -\oint_C f\nabla g \, d\mathbf{r}$$

for any simple closed curve C.

- Moving everything to one side, we see that our goal is to show that

$$\oint_C \big(g\nabla f + f\nabla g\big)\, d\mathbf{r} = 0.$$

This can be addressed in at least two ways:
1. If $h = fg$ then by the product rule, $\nabla h = g\nabla f + f\nabla g$. Thus we obtain the formula above from the fundamental theorem for line integrals, or from the special case of that theorem, which says that the line integral of a conservative field around a closed loop is zero.
2. Applying Greens theorem with

$$\begin{pmatrix} P \\ Q \end{pmatrix} = g\nabla f + f\nabla g = \begin{pmatrix} g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x} \\ g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y} \end{pmatrix}$$

Indeed, a little computation shows $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

(3) From Section 17.6: 20, 24, 42, 46, 60(a)(c).
(In 5th Ed: 18, 22, 38, 44, 56(a)(c).)

(4) Approximating the earth by a sphere, find the latitude above which lies one quarter of the earth's surface. (Note that Los Angeles lies in this 'northern quarter' of the earth.)

- The answer is 30° or equivalently, $\frac{\pi}{6}$ radians:
  The area of the unit sphere is 4π so first we seek ϕ_0 so that

$$\int_0^{\phi_0} \int_0^{2\pi} \sin(\phi) d\theta \, d\phi = \pi.$$

Performing the integrals, this becomes

$$2\pi[1 - \cos(\phi_0)] = \pi$$

which leads to $\phi_0 = \frac{\pi}{3}$ or 60°. However, our parametrization is based on *colatitude*, that is, the angle is measured down from the north pole, rather than up from the equator. Thus, the answer is 30° = 90° − 60°.
  (Incidentally, the latitude of Los Angeles is approximately 34°.)

(5) From Section 17.7: 6, 10, 20, 23.
(In 5th Ed: 6, 10, 28, 23.)

- Problem 17.7.23. The answer is

$$\int_0^{\pi/2} \int_0^{\pi/2} -8\cos^2(\theta)\sin^3(\phi)\, d\theta \, d\phi = -\frac{4\pi}{3}$$