32B Killip Homework 5 Due May 8

(1) From Section 17.1, problems 4, 16, 24 (both editions).

- 17.1.24: The gradient of $f = x\cos(y/z)$ is

$$\nabla f = \begin{pmatrix} \cos(y/z) \\ -\frac{x}{z}\sin(y/z) \\ \frac{xy}{z^2}\sin(y/z) \end{pmatrix}$$

(2) Compute the gradient field of $\frac{1}{4}(x^2+y^2)$ and sketch it.

(3) Consider the curve $\big(x(t), y(t), z(t)\big)$ in space as t varies over $[0,T]$. We could also parameterize this curve by

$$\big(x(\tau^2), y(\tau^2), z(\tau^2)\big) \qquad \tau \in [0, \sqrt{T}].$$

Show that one obtains the same value for the line integral $\int_C f\,ds$ using either parameterization.

- From the definition of line integral, we see that using the first parameterization

$$\int_C f\,ds = \int_0^T f\Big(x(t), y(t), z(t)\Big)\sqrt{\Big[\frac{d}{dt}x(t)\Big]^2 + \Big[\frac{d}{dt}y(t)\Big]^2 + \Big[\frac{d}{dt}z(t)\Big]^2}\;dt$$

Now we look at the second parameterization. By the chain rule,

$$\frac{d}{d\tau}x(\tau^2) = 2\tau\frac{dx}{dt}(\tau^2)$$

and so

$$\sqrt{\Big[\frac{d}{d\tau}x(\tau^2)\Big]^2 + \Big[\frac{d}{d\tau}y(\tau^2)\Big]^2 + \Big[\frac{d}{d\tau}z(\tau^2)\Big]^2} = 2\tau\sqrt{\Big[\frac{dx}{dt}(\tau^2)\Big]^2 + \Big[\frac{dy}{dt}(\tau^2)\Big]^2 + \Big[\frac{dz}{dt}(\tau^2)\Big]^2}$$

Thus we obtain that with the second parameterization,

$$\begin{aligned}\int_C f\,ds &= \int_0^{\sqrt{T}} f\Big(x(\tau^2), y(\tau^2), z(\tau^2)\Big)\sqrt{\Big[\frac{d}{d\tau}x(\tau^2)\Big]^2 + \Big[\frac{d}{d\tau}y(\tau^2)\Big]^2 + \Big[\frac{d}{d\tau}z(\tau^2)\Big]^2}\quad d\tau \\ &= \int_0^{\sqrt{T}} f\Big(x(\tau^2), y(\tau^2), z(\tau^2)\Big)\sqrt{\Big[\frac{dx}{dt}(\tau^2)\Big]^2 + \Big[\frac{dy}{dt}(\tau^2)\Big]^2 + \Big[\frac{dz}{dt}(\tau^2)\Big]^2}\quad 2\tau\,d\tau \\ &= \int_0^T f\Big(x(t), y(t), z(t)\Big)\sqrt{\Big[\frac{d}{dt}x(t)\Big]^2 + \Big[\frac{d}{dt}y(t)\Big]^2 + \Big[\frac{d}{dt}z(t)\Big]^2}\;dt\end{aligned}$$

where we used the change of variables $t = \tau^2$ in the last step. This shows that the two parameterizations give the same answer.

(4) From Section 17.2: 6, 12, 14, 30(a), 40.
(In 5th Ed, these are problems 6, 12, 14, 26(a), 38)

- 17.2.30(a)

$$\begin{aligned} & \int_{-1}^{1} \begin{pmatrix} 2t \\ t^2 \\ 3t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -2t \end{pmatrix} dt \\ = & \int_{-1}^{1} 4t + 3t^2 - 6t^2 \, dt \\ = & -2 \end{aligned}$$