Partial Solutions, Homework 2

(5) Homogeneous means that the density is constant over the triangle. For concreteness, we may as well set this constant to one:

$$m = \int_0^b \int_{h_1(y)}^{h_2(y)} 1 \, dx \, dy$$

where h_1, h_2 are the lines

$$h_1(y) = \tfrac{a}{b} y, \qquad h_2(y) = \tfrac{(a-1)}{b} y + 1$$

Computing the above integral gives

$$m = \int_0^b 1 - \tfrac{y}{b} \, dy = \tfrac{b}{2}.$$

The computations of $\bar{x}$ and $\bar{y}$ proceed similarly:

$$\begin{aligned}
\bar{x} &= \tfrac{1}{m} \int_0^b \int_{h_1(y)}^{h_2(y)} x \, dx \, dy \\
&= \tfrac{2}{b} \int_0^b \tfrac{(b-y)(b+2ay-y)}{2b^2} \, dy = \tfrac{(a+1)}{3} \\
\bar{y} &= \tfrac{1}{m} \int_0^b \int_{h_1(y)}^{h_2(y)} y \, dx \, dy \\
&= \tfrac{2}{b} \int_0^b (1 - \tfrac{y}{b}) y \, dy = \tfrac{b}{3}
\end{aligned}$$

To see that this point lies at the intersection of the medians, we first find the equations of these lines.

The median passing through $(\frac{1}{2}, 0)$ and (a, b) is

$$bx - (a - \tfrac{1}{2}) y = \tfrac{b}{2}$$

It is easy to check that $(\frac{(a+1)}{3}, \frac{b}{3})$ satisfies this equation. The same is true for the other two medians:

$$\begin{aligned}
bx - (a+1)y &= 0 \qquad \text{through } (0,0) \text{ and } (\tfrac{a+1}{2}, \tfrac{b}{2}) \\
bx - (a-2)y &= b \qquad \text{through } (1,0) \text{ and } (\tfrac{a}{2}, \tfrac{b}{2}).
\end{aligned}$$

(6) Determine the center of mass of the homogeneous sector $0 \le \theta \le \pi/6$, $0 \le r \le 1$.

$$\begin{aligned}
m &= \int_0^{\pi/6} \int_0^1 r dr d\theta = \tfrac{\pi}{12} \\
\bar{x} &= \tfrac{1}{m} \int_0^{\pi/6} \int_0^1 r\cos(\theta)\, r dr d\theta \\
&= \tfrac{12}{\pi} \times \tfrac{1}{2} \times \tfrac{1}{3} = \tfrac{2}{\pi} \\
\bar{y} &= \tfrac{1}{m} \int_0^{\pi/6} \int_0^1 r\sin(\theta)\, r dr d\theta \\
&= \tfrac{12}{\pi} \times \tfrac{1}{3} \times (1 - \tfrac{\sqrt{3}}{2}) \\
&= \tfrac{4-2\sqrt{3}}{\pi}
\end{aligned}$$

Determine the moment of inertia of the sector for rotations about the axis passing though the center of mass and perpendicular to the plane of the triangle.

By expanding out the squares,

$$\begin{aligned}
I &= \int_0^{\pi/6} \int_0^1 [r\cos(\theta) - \bar{x}]^2 + [r\sin(\theta) - \bar{y}]^2\, r dr d\theta \\
&= \int_0^{\pi/6} \int_0^1 \left(r^2 - 2\bar{x} r\cos(\theta) - 2\bar{y} r\sin(\theta) + \bar{x}^2 + \bar{y}^2 \right) r dr d\theta
\end{aligned}$$

Thus, recognizing some of the integrals done already in the first part,

$$\begin{aligned}
I &= \int_0^{\pi/6} \int_0^1 r^3\, dr\, d\theta - 2m(\bar{x}^2 + \bar{y}^2) + m(\bar{x}^2 + \bar{y}^2) \\
&= \tfrac{\pi}{24} - m(\bar{x}^2 + \bar{y}^2) \\
&= \tfrac{\pi}{24} - \tfrac{\pi}{12}\left(\tfrac{4}{\pi^2} + \tfrac{(4-2\sqrt{3})^2}{\pi^2}\right) \\
&= \tfrac{\pi}{24} - \tfrac{8}{3\pi} + \tfrac{4}{\pi\sqrt{3}}
\end{aligned}$$

(7) Using the elementary properties of integrals,

$$\begin{aligned}
I(x_1, y_1) &= \iint_D \left[(x - x_1)^2 + (y - y_1)^2\right] \rho(x, y)\, dA \\
&= \iint_D \left[(x - \bar{x} + \bar{x} - x_1)^2 + (y - \bar{y} + \bar{y} - y_1)^2\right] \rho(x, y)\, dA \\
&= \iint_D \left[(x - \bar{x})^2 + 2(x - \bar{x})(\bar{x} - x_1) + (\bar{x} - x_1)^2\right] \rho(x, y)\, dA \\
&\qquad + \iint_D \left[(y - \bar{y})^2 + 2(y - \bar{y})(\bar{y} - y_1) + (\bar{y} - y_1)^2\right] \rho(x, y)\, dA \\
&= \iint_D \left[(x - \bar{x})^2 + (y - \bar{y})^2\right] \rho(x, y)\, dA \\
&\qquad + 2(\bar{x} - x_1) \iint_D (x - \bar{x})\, \rho(x, y)\, dA \\
&\qquad + 2(\bar{y} - y_1) \iint_D (y - \bar{y})\, \rho(x, y)\, dA \\
&\qquad + \left[(\bar{x} - x_1)^2 + (\bar{y} - y_1)^2\right] \iint_D \rho(x, y)\, dA
\end{aligned}$$

Looking at the last formula (which is split over four lines), we see that the first line is $I(\bar{x}, \bar{y})$, and that the last line is $\left[(\bar{x} - x_1)^2 + (\bar{y} - y_1)^2\right] m$. Thus we will be finished once we show that the integrals on the middle two lines are zero. This relies on the definition of $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{1}{m} \iint_D x\, \rho(x, y)\, dA$$

which implies (after multiplying both sides by $m = \iint_D \rho(x, y)\, dA$),

$$\iint_D \bar{x}\, \rho(x, y)\, dA = \iint_D x\, \rho(x, y)\, dA.$$

This kills the second line; doing the same for $\bar{y}$, kills the third.