

(1) Determine the surface area of the paraboloid

$$x^2 + y^2 = 2z, \quad 0 \leq z \leq 1$$

by whatever means you wish.

Inspired by cylindrical co-ords, we parameterize the surface by

$$x = u \cos(v)$$

$$y = u \sin(v)$$

$$z = \frac{1}{2}u^2$$

$$\text{i.e., } \tilde{\Gamma}(u,v) = \begin{pmatrix} u \cos(v) \\ u \sin(v) \\ \frac{1}{2}u^2 \end{pmatrix}$$

with  $0 \leq u \leq \sqrt{2}$  and  $0 \leq v \leq 2\pi$ .

$$\tilde{\Gamma}_u \times \tilde{\Gamma}_v = \begin{pmatrix} \cos(v) \\ \sin(v) \\ u \end{pmatrix} \times \begin{pmatrix} -u \sin(v) \\ u \cos(v) \\ 0 \end{pmatrix} = \begin{pmatrix} -u^2 \cos(v) \\ -u^2 \sin(v) \\ u \end{pmatrix}$$

$$\text{So } \|\tilde{\Gamma}_u \times \tilde{\Gamma}_v\| = \sqrt{u^4 \cos^2(v) + u^4 \sin^2(v) + u^2} = u \sqrt{u^2 + 1}.$$

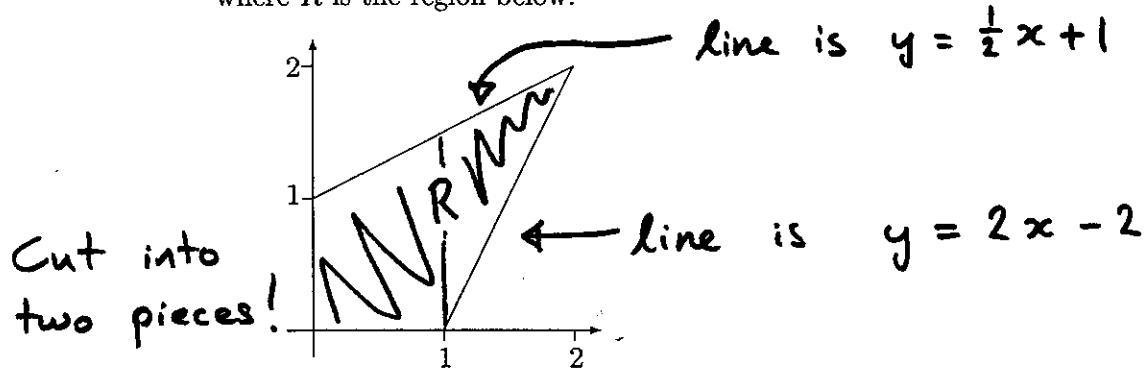
and

$$\begin{aligned} \text{Area} &= \iint_{\text{Paraboloid}} 1 \, dS = \int_0^{\sqrt{2}} \int_0^{2\pi} u \sqrt{u^2 + 1} \, dv \, du \\ &= 2\pi \int_0^{\sqrt{2}} u \sqrt{u^2 + 1} \, du \stackrel{\begin{array}{l} t = u^2 + 1 \\ dt = 2u \, du \end{array}}{=} \pi \int_1^3 \sqrt{t} \, dt \\ &= \frac{2\pi}{3} \left[ t^{\frac{3}{2}} \right]_1^3 \\ &= \frac{2\pi}{3} \{ 3^{\frac{3}{2}} - 1 \}. \end{aligned}$$

(2) Calculate the integral

$$\int_R x^2 dA$$

where  $R$  is the region below:



$$\begin{aligned} \text{Ans} &= \int_0^1 \int_0^{\frac{1}{2}x+1} x^2 dy dx + \int_1^2 \int_{2x-2}^{\frac{1}{2}x+1} x^2 dy dx \\ &= \int_0^1 \frac{1}{2}x^3 + x^2 + \int_1^2 [\frac{1}{2}x+1 - (2x-2)] x^2 dx \\ &= \frac{1}{8} + \frac{1}{3} + \int_1^2 -\frac{3}{2}x^3 + 3x^2 dx \\ &= \frac{1}{8} + \frac{1}{3} + \left[ -\frac{3}{8}x^4 + x^3 \right]_1^2 \\ &= \frac{1}{8} + \frac{1}{3} - 6 + \frac{3}{8} + 8 - 1 \\ &= 1 + \frac{1}{2} + \frac{1}{3} \\ &= 1 + \frac{5}{6} = \frac{11}{6} \end{aligned}$$

(3) State the Fundamental Theorem for Line integrals

Given a function  $f$  defined on the plane  
(or in three-space), and a curve  $C$   
from  $\underline{z}_0$  to  $\underline{z}_1$ , then

$$\int_C \nabla f \cdot d\underline{r} = f(\underline{z}_1) - f(\underline{z}_0)$$

(4)+(5) Compute both sides of the Divergence Theorem for the cylinder

$$x^2 + y^2 \leq 1 \quad 0 \leq z \leq 1$$

with  $\mathbf{F} = z \sin(x^2 + y^2) \mathbf{k}$ . (Of course, they should turn out to be equal).

$$\iiint \nabla \cdot \mathbf{F} dV = \iint \mathbf{F} \cdot d\mathbf{S}$$

**LHS**  $\nabla \cdot \mathbf{F} = \sin(x^2 + y^2)$

Using cylindrical co-ords,

$$\text{LHS} = \int_0^1 \int_0^1 \int_0^{2\pi} \sin(r^2) r d\theta dr dz$$

$$= \int_0^1 \int_0^1 2\pi r \sin(r^2) dr dz \quad \downarrow t = r^2 \quad dt = 2rdr$$

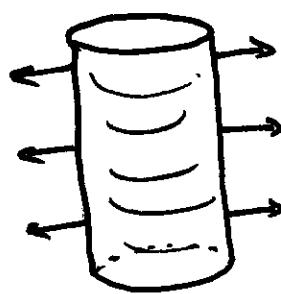
$$= \int_0^1 \int_0^1 \pi \sin(t) dt dz$$

$$= \int_0^1 \pi [1 - \cos(1)] dz$$

$$= \pi [1 - \cos(1)]$$



**RHS** The surface of the cylinder comprises three pieces



[Extra space for (4)+(5)]

Top

$$\begin{aligned}x &= u \cos(v) \\y &= u \sin(v) \\z &= 1\end{aligned}$$

$$\underline{r}_u \times \underline{r}_v = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 0 \end{pmatrix} \times \begin{pmatrix} -u \sin(v) \\ u \cos(v) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

$$\text{contribution} = \int_0^1 \int_0^{2\pi} \begin{pmatrix} 0 \\ 0 \\ \sin(u^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} du dv$$

$$= \int_0^1 2\pi u \sin(u^2) du = \pi [1 - \cos(1)] \quad \text{cf. calculation of LHS}$$

Bottom contributes zero since  $z=0 \Rightarrow \underline{F}=0$

Round surface contributes zero since  $\underline{F}$  points along the surface not through it:  $\underline{n} \cdot \underline{F} = 0$ .

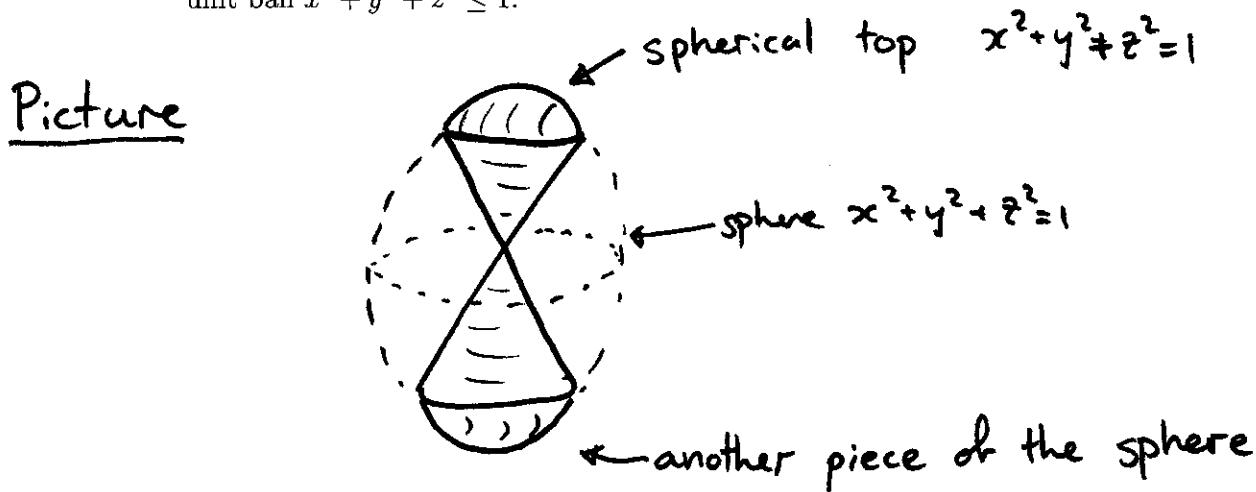
Conclusion:

$$\begin{aligned}\text{RHS} &= \pi[1 - \cos(1)] + 0 + 0 \\&= \pi[1 - \cos(1)] \\&= \text{LHS.}\end{aligned}$$

(6) Determine

$$\int_R z dV$$

where  $R$  is the intersection of the cone  $x^2 + y^2 \leq z^2$  and the unit ball  $x^2 + y^2 + z^2 \leq 1$ .



Choose spherical co-ordinates:

Being inside the sphere means  $* 0 \leq \rho \leq 1$

Being inside the cone means  $0 \leq \varphi \leq \frac{\pi}{4}$

OR  $\frac{3\pi}{4} \leq \varphi \leq \pi$

Cylindrical symmetry  $\rightarrow 0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} \iiint_{\text{top part}} z dV &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 g \cos(\varphi) g^2 \sin(\varphi) dg d\theta d\varphi \\ &= \cancel{\frac{\pi}{2}} \int_0^{\pi/4} \cos(\varphi) \sin(\varphi) d\varphi = \frac{\pi}{4} \cdot \int_0^{\pi/4} \sin(2\varphi) d\varphi \\ &= \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} \iiint_{\text{bottom part}} z dV &= \dots \underset{\text{as above}}{=} \frac{\pi}{4} \int_{3\pi/4}^{\pi} \sin(2\varphi) d\varphi = -\frac{\pi}{8} \end{aligned}$$

$$\text{Final Answer} = \frac{\pi}{8} - \frac{\pi}{8} = 0.$$

(7) Justify the following statement: If  $\nabla \times \mathbf{F} = 0$  then

$$\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = 0$$

for any closed loop  $\gamma$ .

Soln #1 If  $\underline{\mathbf{F}}$  is defined everywhere (or more generally on a simply connected region) and  $\nabla \times \underline{\mathbf{F}} = 0$  then  $\underline{\mathbf{F}} = \nabla f$  for some function  $f$ . But then by the Fundamental Theorem for Line Integrals,

$$\oint_{\gamma} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \oint_{\gamma} \nabla f \cdot d\underline{\mathbf{r}} = f(\text{end of } \gamma) - f(\text{beginning of } \gamma) \\ = 0$$

since a closed loop begins & ends at the same place.

Soln #2 Suppose  $\underline{\mathbf{F}}$  is defined everywhere\* and let  $S$  be a surface having  $\gamma$  as its boundary (oriented appropriately).

By Stokes' Theorem,  $\oint_{\gamma} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \iint_S (\nabla \times \underline{\mathbf{F}}) \cdot d\underline{\mathbf{S}} = 0$ .

\* Actually, for such a surface to exist, it suffices that  $\underline{\mathbf{F}}$  be defined in a simply-connected region: the ability to contract a loop to a point is equivalent to the possibility of 'filling in the hole' with a surface.

(8) Consider the region  $R$  given by

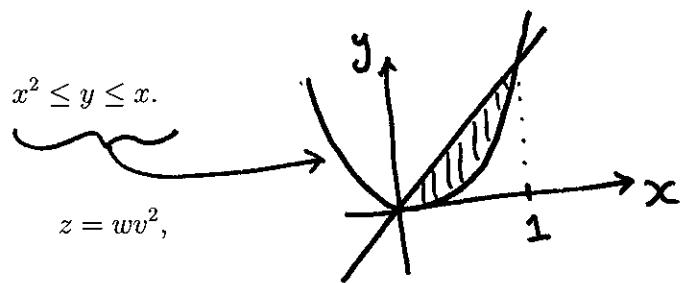
$$0 \leq z \leq (y - x^2)^2, \quad x^2 \leq y \leq x.$$

Use the change of variables

$$x = u, \quad y = v + u^2, \quad z = wv^2,$$

to evaluate

$$\int_R \frac{dV}{y - x^2}$$



$$\text{Jacobian} = \begin{vmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2vw & v^2 \end{vmatrix} = v^2 \text{ clearly +ve.}$$

Equations defining region in new co-ords:

$$0 \leq wv^2 \leq v^2 \quad \& \quad u^2 \leq v + u^2 \leq u$$

$$\text{i.e. } 0 \leq w \leq 1 \quad \& \quad 0 \leq v \leq u - u^2$$

note  $0 \leq u - u^2$  implies  $0 \leq u \leq 1$  cf. picture above.

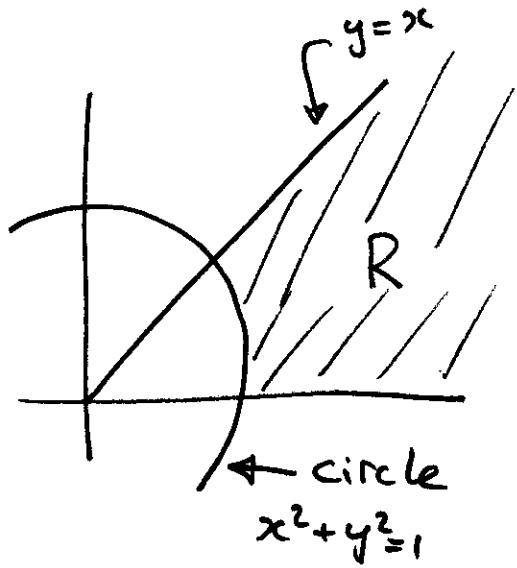
$$\begin{aligned} \text{Ans} &= \int_0^1 \int_0^{u-u^2} \int_0^1 \frac{v^2}{v} dw dv du \\ &= \int_0^1 \left[ \frac{1}{2} v^2 \right]_0^{u(1-u)} du = \frac{1}{2} \int_0^1 u^2(1-u)^2 \\ &= \frac{1}{2} \int_0^1 u^2 - 2u^3 + u^4 du \\ &= \frac{1}{2} \left\{ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right\} = \frac{1}{60} \end{aligned}$$

(9) Let  $R$  be the region where  $0 \leq y \leq x$  and  $x^2 + y^2 \geq 1$ . Evaluate

$$\int_R \frac{dA}{(x^2 + y^2)^2}$$

by switching to polar coordinates.

$$\begin{aligned} \text{Ans} &= \int_0^{\pi/4} \int_1^\infty \frac{r dr d\theta}{r^4} \\ &= \int_0^{\pi/4} \left[ -\frac{1}{2} r^{-2} \right]_1^\infty d\theta \\ &= \int_0^{\pi/4} \frac{1}{2} d\theta \\ &= \frac{\pi}{8} \end{aligned}$$



(10) Which of the following is conservative:

$$\underline{F}_1 = x\mathbf{i} + e^y\mathbf{j} + xe^y\mathbf{k} \quad \text{or} \quad \underline{F}_2 = ye^x\mathbf{i} + e^x\mathbf{j} + z\mathbf{k}$$

Write it as  $\nabla f$ .

$$\nabla \times \underline{F}_1 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & e^y & xe^y \end{vmatrix} = (xe^y - 0)\hat{i} + \dots \neq 0$$

$$\nabla \times \underline{F}_2 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & e^x & z \end{vmatrix} = (0 - 0)\hat{i} + (0 - 0)\hat{j} + (e^x - e^x)\hat{k} = 0$$

$\underline{F}_2$  is defined everywhere &  $\nabla \times \underline{F}_2 = 0$ . Therefore,  $\underline{F}_2$  is conservative.

Now we seek  $f$ :

$$\text{As } \frac{\partial f}{\partial x} = ye^x, \quad f = ye^x + g(y, z)$$

$$\text{but then } e^x = \frac{\partial f}{\partial y} = e^x + \frac{\partial g}{\partial y} \quad \text{so} \quad \frac{\partial g}{\partial y} = 0 \quad \& \quad f = ye^x + h(z)$$

$$\text{Next } z = \frac{\partial f}{\partial z} = 0 + h'(z) \quad \text{so} \quad h'(z) = z \quad \& \quad h(z) = \frac{1}{2}z^2 + C$$

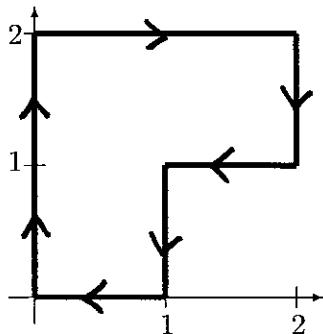
Therefore, the most general potential function is

$$f = ye^x + \frac{1}{2}z^2 + C, \quad C \text{ a constant.}$$

(11) Evaluate

$$\int_C \cos(\pi x) dx + x dy$$

where  $C$  is the curve



$$P = \cos(\pi x)$$

$$Q = x$$

Use Green's Theorem ('cause it's quicker)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$$

To correct for orientation of  $C$

$$\oint_C \cos(\pi x) dx + x dy = - \iint_D 1 dA$$

$$= -\text{Area}(D)$$

$$= -3$$

(12) Determine

$$\int \frac{dS}{\sqrt{x^2 + z^2}}$$

over the oblique cone parameterized by

$$x = v + v \cos(u), \quad y = v \sin(u), \quad z = v$$

for  $u \in [0, 2\pi]$  and  $v \in [0, 1]$ .

$$\Gamma_u \times \Gamma_v = \begin{pmatrix} -v \sin(u) \\ v \cos(u) \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 + \cos(u) \\ \sin(u) \\ 1 \end{pmatrix} = \begin{pmatrix} v \cos(u) \\ v \sin(u) \\ -v[1 + \cos(u)] \end{pmatrix}$$

$$\begin{aligned} \text{So } \|\Gamma_u \times \Gamma_v\| &= \sqrt{v^2 + v^2[1 + \cos(u)]^2} \\ &= v \cdot \sqrt{2 + 2\cos(u) + \cos^2(u)} \end{aligned}$$

Also,

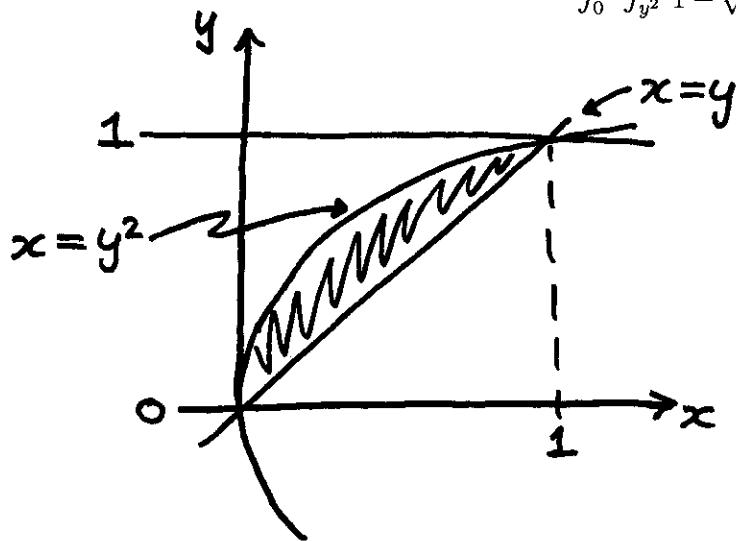
$$\sqrt{x^2 + z^2} = v \sqrt{[1 + \cos(u)]^2 + 1^2} = v \sqrt{2 + 2\cos(u) + \cos^2(u)}$$

Therefore,

$$\text{Ans} = \int_0^{2\pi} \int_0^1 1 \, dv \, du = 2\pi$$

(13) Compute the following integral by reversing the order

$$\int_0^1 \int_{y^2}^y \frac{1}{1-\sqrt{x}} dx dy$$



$$Ans = \int_0^1 \int_{\sqrt{x}}^{\sqrt{x'}} \frac{1}{1-\sqrt{x}} dy dx = \int_0^1 \frac{\sqrt{x'} - x}{1 - \sqrt{x}} dx$$

$$\text{but } \sqrt{x'} - x = \sqrt{x} \cdot (1 - \sqrt{x}) \quad \text{so}$$

$$Ans = \int_0^1 \sqrt{x} dx = \left[ \frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3}.$$