271D Homework.

1. Proceeding directly from the definition, show that the Laplacian defined as follows,
$$D(-\Delta) = \{u \in L^2 : \int |\xi|^4 |\hat{u}(\xi)|^2 < \infty\} \quad \text{and} \quad -\widehat{\Delta u}(\xi) = |\xi|^2 \hat{u}(\xi)$$
is a self-adjoint operator on $L^2(\mathbb{R}^3)$. Our convention for the Fourier transform is as follows:
$$\hat{u}(\xi) = (2\pi)^{-3/2} \int e^{-i\xi\cdot x} u(x)\, dx$$
for all $u \in L^1 \cap L^2$ and then extended to $u \in L^2$ by continuity.

2. What is the dimension of the Fermionic Fock space based on $\mathcal{H} = \mathbb{C}^k$ as the one-particle space.

3. Fix $N \geq 2$ and define
$$E(\psi) = \sum_i \int |\nabla_i \psi(\vec{x})|^2 \, dx_1 \cdots dx_N + \sum_{i<j} \int V(x_i - x_j) |\psi(\vec{x})|^2 \, dx_1 \cdots dx_N$$
for all $\psi \in H^1(\mathbb{R}^{3N}/L\mathbb{Z}^{3N})$. Show that the infimum of $E(\psi)$ over all L^2-normalized vectors ψ is achieved. (We impose no symmetry constraint.)