Lemma 1 (Riesz Lemma)**.** *Fix $0 < \theta < 1$. If $M \subsetneq X$ is a proper closed subspace of a Banach space X then one can find $x \in X$ with $\|x\| = 1$ and $\operatorname{dist}(x, M) \geq \theta$.*

Proof. By the hyperplane separation theorem, there is a unit element $\ell \in X^*$ that vanishes on M. Now choose x so that $\ell(x) \geq \theta$. As ℓ is 1-Lipschitz, $|\ell(x)| \leq \operatorname{dist}(x, M)$. $\square$

By employing this lemma inductively, we obtain the following, which better reflects how we will use the lemma.

Corollary 2 (Riesz Lemma)**.** *Given a strictly nested sequence of closed subspaces*

$$\{0\} \subsetneq N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq N_4 \subsetneq \cdots$$

of a Banach space X, one can find a sequence of vectors $x_n \in N_n$ with $\|x_n\| = 1$ and $\operatorname{dist}(x_n, N_{n-1}) \geq \frac{1}{2}$. Similarly, for a sequence of closed subspaces nested in the opposite direction, $R_1 \supsetneq R_2 \supsetneq \cdots$, there are unit vectors $x_n \in R_n$ with $\operatorname{dist}(x_n, R_{n+1}) > \frac{1}{2}$.

Proposition 3. *Suppose $T : X \to X$ is compact and $\lambda \neq 0$.*

(a) *For each integer $m \geq 1$, $\ker(\lambda - T)^m$ is finite dimensional. Moreover, there is an integer k so that $\ker(\lambda - T)^m \subseteq \ker(\lambda - T)^k$ for every integer $m \geq 1$.*
(b) *The range of $(\lambda - T)^m$ is closed for each integer $m \geq 1$.*
(c) *$\lambda - T$ is surjective $\iff$ it is injective.*

Remarks. 1. When $\ker(\lambda - T)$ is nontrivial, then λ is an eigenvalue of T. In this case, $\dim \ker(\lambda - T)$ is called the *geometric mulitplicity* of the eigenvalue, while $\dim \ker(\lambda - T)^k$ (with k as in (a)) is called the *algebraic multiplicity.*

2. As $\ker(\lambda - T)^m \subseteq \ker(\lambda - T)^{k+1}$, so $\dim \ker(\lambda - T)^{m+1} \leq \dim \ker(\lambda - T)^{m+1}$. However, the rate at which the dimension increases is decreasing:

$$\dim \ker(\lambda - T)^{m+2} - \dim \ker(\lambda - T)^{m+1} \leq \dim \ker(\lambda - T)^{m+1} - \dim \ker(\lambda - T)^m \tag{1}$$

To see this, argue that $\lambda - T$ defines an injective map from $\ker(\lambda - T)^{m+2} / \ker(\lambda - T)^{m+1}$ to $\ker(\lambda - T)^{m+1} / \ker(\lambda - T)^m$.

Proof. By rescaling $T \mapsto \lambda^{-1}T$, it suffices to consider the case $\lambda = 1$.

If the kernel of $1 - T$ were infinite dimensional, then by the Riesz Lemma we can find a $\frac{1}{2}$-separated sequence of unit vectors therein. But T is compact, so $x_n = Tx_n$ lie in a compact set, which contradicts their separation.

As T is compact, so is

$$1 - (1 - T)^m = \tbinom{m}{1}T - \tbinom{m}{2}T^2 \pm \cdots \pm T^m. \tag{2}$$

Thus the reasoning just given shows that the kernel of $(1 - T)^m$ is also finite dimensional.

If the nested sequence of subspaces $N_m := \ker(\lambda - T)^m$ did not stabilize then by (1) we see that each is *properly* contained in its successor. Thus we may apply the Riesz lemma to produce a sequence $x_m \in N_m$ with $\operatorname{dist}(x_m, N_{m-1}) \geq \frac{1}{2}$. But then for $n > m$,

$$Tx_n - Tx_m = x_n - (1 - T)x_n - x_m + (1 - T)x_m \in x_n + N_{n-1}.$$

Thus $\|Tx_n - Tx_m\| \geq \operatorname{dist}(x_n, N_{n-1}) \geq \frac{1}{2}$ which implies the image of the unit ball under T contains a $\frac{1}{2}$-separated sequence. This contradicts the compactness of T.

We now turn to part (b). By the trick (2), it suffices to treat the case $m = 1$. This requires us to show that if $y_n = (1 - T)x_n$ and $y_n \to y$ then there exists $x \in X$ so that $y = (1 - T)x$. For brevity, we continue to use the notation $N_1 := \ker(1 - T)$.

First we claim that $d_n := \operatorname{dist}(x_n, N_1)$ is bounded. We prove this by contradiction and so assume (perhaps after passing to a subsequence) that $d_n \to \infty$. Choose $z_n \in N_1$ so that $\|x_n - z_n\| < 2d_n$ and observe that

$$(1-T)d_n^{-1}(x_n - z_n) = d_n^{-1}y_n \to 0. \tag{3}$$

As T is compact, any subsequence of $d_n^{-1}(x_n - z_n)$ has a subsequence so that $d_n^{-1}T(x_n - z_n)$ converges. In view of (3) above, $d_n^{-1}(x_n - z_n)$ also converges to some $w \in X$ along the subsequence; indeed we see that $w \in N_1$. But this leads us to a contradiction:

$$1 = d_n^{-1} \operatorname{dist}(x_n, N_1) \leq d_n^{-1}\|x_n - (z_n - w)\| = \|d_n^{-1}(x_n - z_n) - w\| \to 0 \quad \text{as } n \to \infty.$$

Having proved our claim, we know that we may choose $z_n \in N_1$ so that $x_n - z_n$ is bounded. From this and the compactness of T, we deduce that $T(x_n - z_n)$ has a convergent subsequence. Moreover, along this subsequence, $(1-T)(x_n - z_n) = y_n \to y$ and so not only does $x_n - z_n$ have a limit, say $x \in X$, but this limit obeys $(1-T)x = y$. This completes the proof of (b).

Consider now the $\Rightarrow$ direction of part (c). Suppose $(1-T)$ is not injective; then there is a non-zero $x_1 \in \ker(1-T)$. But if $(1-T)$ were surjective then there would be an $x_2 \in X$ so that $(1-T)x_2 = x_1$. Proceeding inductively we find a sequence of linearly independent vectors x_n so that $(1-T)^n x_n = 0$. This contradicts part (a) of the current Theorem.

To prove the other implication of part (c) we suppose $(1-T)$ is injective, but not surjective. Then $R_m := (1-T)^m X$ form a properly nested sequence of closed (cf. part (b)) subspaces. Choosing $x_n \in R_n$ as in Corollary 2 we find that for $n < m$,

$$Tx_n - Tx_n = x_n - (1-T)x_n - x_m + (1-T)x_m \in x_n + R_{n+1}$$

and so $\|Tx_n - Tx_m\| \geq \operatorname{dist}(x_n, R_{n+1}) \geq \frac{1}{2}$. This contradicts the compactness of T. $\square$

Theorem 4. *Suppose $T : X \to X$ is compact, then*
(a) *Every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue with finite geometric and algebraic multiplicities.*
(b) *If X is infinite dimensional, $0 \in \sigma(T)$.*
(c) *$\sigma(T)$ is countable and may only accumulate at 0.*

Proof. (a) If $0 \neq \lambda \in \mathbb{C}$ is not an eigenvalue then $(\lambda - T)$ is injective. By Proposition 3(c) it must also be surjective. The open mapping theorem then implies that $(\lambda - T)$ is continuously invertible, which shows that $\lambda \notin \sigma(T)$.

(b) Let B denote the closed unit ball in X. If $0 \notin \sigma(T)$ then T is (continuously) invertible and so writing $B = T^{-1}TB$ we see that B lies in the continuous image of a compact set, namely, $\overline{TB}$. This implies that the unit ball is (norm-)compact, which is only true when X is finite dimensional (cf. the Riesz lemma).

(c) As $\sigma(T)$ is compact, it suffices to show that there are no non-zero accumulation points. Suppose to the contrary that there is a sequence $\lambda_n \in \sigma(T)$ with $|\lambda_n| > \delta > 0$ for all n. Then we apply Corollary 2 to the combined eigen-spaces

$$N_n := \ker(\lambda_n - T) + \cdots + \ker(\lambda_1 - T)$$

to find $x_n \in N_n$ with $\operatorname{dist}(x_n, N_{n-1}) > \frac{1}{2}$. Noting that $(\lambda_n - T) : N_n \to N_{n-1}$ we find

$$Tx_n - Tx_m = \lambda_n x_n - (\lambda_n - T)x_n - \lambda_m x_m \in \lambda_n x_n + N_{n-1}$$

for any $n > m$. Thus $\|Tx_n - Tx_m\| \geq |\lambda_n| \operatorname{dist}(x_n, N_{n-1}) \geq \frac{\delta}{2}$, which contradicts the compactness of T. $\square$

Theorem 5. *(Fredholm Alternative I) If $T : X \to X$ is compact then*

$$(1 - T)X = \ker(1 - T')^{\top} := \{x \in X : \ell(x) = 0 \text{ for all } \ell \in \ker(1 - T')\}$$

Equivalently, given $y \in X$ there exists $x \in X$ with $(1 - T)x = y$ if and only if for every $\ell \in X^*$, $(1 - T')\ell = 0$ implies $\ell(y) = 0$.

Remark. We use the upside-down $\perp$ symbol to distinguish from the annihilating set in X^{**}.

Proof. The key point is that the range of $(1 - T)$ is closed. Recalling a few definitions,

$$\begin{aligned} \ell \in \ker(1 - T') &\iff \ell \circ (1 - T) = 0 \quad (\text{as an element of } X^*) \\ &\iff (1 - T)X \subseteq \ker \ell. \end{aligned}$$

That is, $\ker(1 - T') = [(1 - T)X]^{\perp}$. Next we recall that by the hyperplane separation theorem, the closure of a vector subspace $M \subseteq X$ can be computed as

$$\bar{M} = (M^{\perp})^{\top} = \{x \in X : \ell(x) = 0 \text{ whenever } M \subseteq \ker \ell\}.$$

By Proposition 3(b), we know that $(1 - T)X$ is closed. □

By combining this theorem with Proposition 3(c) we obtain the following variant:

Corollary 6. *(Fredholm Alternative II) If $T : X \to X$ is compact then*

$$1 - T \text{ is invertible} \iff 1 - T' \text{ is invertible}$$

and in particular, $\sigma(T) = \sigma(T')$.

To see the 'alternative' in the Fredholm Alternative, we note the following:

Corollary 7. *(Fredholm Alternative III) If $T : X \to X$ is compact and then either*
(a) *$x - Tx = y$ has a solution for all $y \in X$; or*
(b) *$\ell - T'\ell = 0$ has a non-zero solution $\ell \in X^*$;*
but never both.