Definition 1. For $1 \leq p < \infty$ and $f : \mathbb{R}^d \to \mathbb{C}$ we define
\[ \|f\|_{L^p_{\text{weak}}(\mathbb{R}^d)} := \sup_{\lambda > 0} \lambda \{x : |f(x)| > \lambda\}^{1/p} \]
and the weak $L^p$ space
\[ L^p_{\text{weak}}(\mathbb{R}^d) := \{f : \|f\|_{L^p_{\text{weak}}(\mathbb{R}^d)} < \infty\}. \]
Equivalently, $f \in L^p_{\text{weak}}$ if and only if $\{x : |f(x)| > \lambda\} \lesssim \lambda^{-p}$.

Warning. The quantity in (1) does not define a norm. This is the reason we append the asterisk to the usual norm notation.

To make a side-by-side comparison with the usual $L^p$ norm, we note that
\[ \|f\|_{L^p} = \left( \int_0^\infty \int_{|f| < \lambda} p\lambda^{p-1} d\lambda dx \right)^{1/p} = \left( \int_0^\infty |\{x : |f(x)| > \lambda\}| p\lambda^{p-1} d\lambda \right)^{1/p} = p^{1/p} \|\lambda|\{x : |f| > \lambda\}|^{1/p}\|_{L^p((0, \infty), \frac{d\lambda}{\lambda})} \]
and, with the convention that $p^{1/\infty} = 1$,
\[ \|f\|_{L^p_{\text{weak}}} = p^{1/\infty} \|\lambda|\{x : |f| > \lambda\}|^{1/p}\|_{L^\infty((0, \infty), \frac{d\lambda}{\lambda})}. \]

This suggests the following definition.

Definition 2. For $1 \leq p < \infty$ and $1 \leq q \leq \infty$ we define the Lorentz space $L^{p,q}(\mathbb{R}^d)$ as the space of measurable functions $f$ for which
\[ \|f\|_{L^{p,q}} := p^{1/q} \|\lambda|\{x : |f| > \lambda\}|^{1/p}\|_{L^q((0, \infty), \frac{d\lambda}{\lambda})} < \infty. \]

From the discussion above, we see that $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_{\text{weak}}$. Again $\|\cdot\|_{L^{p,q}}$ is not a norm in general. Nevertheless, it is positively homogeneous: for all $a \in \mathbb{C}$,
\[ \|af\|_{L^{p,q}} = \|\lambda|\{x : |f| > \lambda^{-1}\}|^{1/p}\|_{L^q((a\lambda, \infty), \frac{d\lambda}{\lambda})} = \|a\| \cdot \|f\|_{L^{p,q}} \]
(strictly the case $a = 0$ should receive separate treatment). In lieu of the triangle inequality, we have the following:
\[ \|f + g\|_{L^{p,q}} = \|\lambda|\{f + g| > \lambda\}|^{1/p}\|_{L^q((\lambda, \infty), \frac{d\lambda}{\lambda})} \leq \|\lambda (|f| > \frac{1}{2}\lambda) + |g| > \frac{1}{2}\lambda\}|^{1/p}\|_{L^q((\lambda, \infty), \frac{d\lambda}{\lambda})} \leq \|\lambda|\{f| > \frac{1}{2}\lambda\}|^{1/p}\|_{L^q((\lambda, \infty), \frac{d\lambda}{\lambda})} + \|\lambda|g| > \frac{1}{2}\lambda\}|^{1/p}\|_{L^q((\lambda, \infty), \frac{d\lambda}{\lambda})} \]
by the subadditivity of fractional powers and the triangle inequality in $L^q((\lambda, \infty), \frac{d\lambda}{\lambda})$. Thus
\[ \|f + g\|_{L^{p,q}} \leq 2\|f\|_{L^{p,q}} + 2\|g\|_{L^{p,q}}. \]

Combining (3), (4), and the fact that $\|f\|_{L^{p,q}} = 0$ implies $f \equiv 0$ almost everywhere, we see that $\|\cdot\|_{L^{p,q}}$ obeys the axioms of a quasi-norm. When $p > 1$, this quasi-norm is equivalent to an actual norm (see below). When $p = 1$ and $q \neq 1$, there cannot be a norm that is equivalent to our quasi-norm. However there is a metric that generates the same topology. In either case, we obtain a complete metric space.

Notice that (i) if $|f| \geq |g|$ then $\|f\|_{L^{p,q}} \geq \|g\|_{L^{p,q}}$ and (ii) The quasi-norms are rearrangement invariant, which is to say that $\|f\|_{L^{p,q}} = \|f \circ \phi\|_{L^{p,q}}$ for any measure preserving bijection $\phi : \mathbb{R}^d \to \mathbb{R}^d$.  

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Proposition 3. Given \( f \in L^{p,q} \), we write \( f = \sum f_m \) where 

\[
    f_m(x) := f(x) \chi_{\{|x-2^m| \leq |f(x)| < 2^{m+1}\}}.
\]

Then 

\[
    \|f\|_{L^{p,q}} \approx_{p,q} \|f_m\|_{L^p(\mathbb{R}^d)} \|f_m\|_{\ell^q_\mathbb{N}(\mathbb{Z})}.
\]

In particular, \( L^{p,q_1} \subseteq L^{p,q_2} \) whenever \( q_1 \leq q_2 \).

Proof. It suffices to consider \( L \) where all sums are over \( A \) where 

\[
    \sum_{n \geq m} |E_n| \leq \frac{q}{p} \sum_{n \geq m} |E_n| \leq \sum_{n \geq m} |E_n| \leq \sum_{n \geq m} |E_n|.
\]

To obtain a lower bound, we keep only the summand \( m = n \); for an upper bound, we use the subadditivity of fractional powers. This yields 

\[
\begin{aligned}
    \|f_m\|_{L^{p,q}}^q &= p \int_0^\infty \lambda^q |\{|f| > \lambda\}|^{q/p} \frac{d\lambda}{\lambda} \\
    &= p \sum_m \int_{2^{m-1}}^{2^m} \lambda^q \left( \sum_{n \geq m} |E_n| \right)^{q/p} \frac{d\lambda}{\lambda} \\
    &\approx \sum_m \left( \sum_{n \geq m} |E_n| \right)^{1/p}.
\end{aligned}
\]

As \( \|2^m \chi_{E_m}\|_{L^p} = 2^m \|E_m\|_{L^p}^{1/p} \), we have our desired lower bound. To obtain the upper bound, we use the triangle inequality in \( \ell^q(\mathbb{Z}) \):

\[
\text{RHS}(5) = \left\| \sum_{k=0}^\infty 2^{-k} \|2^{m+k} \chi_{E_{m+k}}\|_{L^p} \right\|_{\ell^q_n} \leq \sum_{k=0}^\infty 2^{-k} \|2^m \chi_{E_m}\|_{L^p} \left\| \sum_{k=0}^\infty 2^{-k} \|2^m \chi_{E_m}\|_{L^p} \right\|_{\ell^q_n}.
\]

This completes the proof of the upper bound. \( \square \)

Lemma 4. Given \( 1 \leq q < \infty \) and a finite set \( A \subset 2^\mathbb{N} \), 

\[
    \sum A^q \leq \left| \sum A \right|^q \leq 2 \max_{A \in A} A \leq 2^q \sum A^q
\]

where all sums are over \( A \in A \). More generally, for any subset \( A \) of a geometric series and any \( 0 < q < \infty \),

\[
    \sum A^q \approx \left| \sum A \right|^q
\]

where the implicit constants depend on \( q \) and the step size of the geometric series.

Proposition 5. For \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \),

\[
\sup \{ \|fg\|_{L^{p,q}} : \|g\|_{L^{p',q'}} \leq 1 \} \approx \|f\|_{L^{p,q}}.
\]

Indeed, \( \text{LHS}(6) \) defines a norm on \( L^{p,q} \). Note that by (6), this norm is equivalent to our quasi-norm. Moreover, under this norm, \( L^{p,q} \) is a Banach space and when \( q \neq \infty \), the dual Banach space is \( L^{p',q'} \), under the natural pairing.

Remark. When \( p = 1 \) (and \( q \neq 1 \)), the LHS(6) is typically infinite; indeed, \( \int_E |f| \) may well be infinite even for some set \( E \) of finite measure. In fact, there there cannot be a norm on \( L^{p,q} \) equivalent to our quasi-norm. For example, the impossibility of finding an equivalent norm for \( L^{1,\infty}(\mathbb{R}) \) can be deduced by computing

\[
\left\| \sum_{n=0}^N |x-n|^{-1} \right\|_{L^{1,\infty}} \approx N \log(N) \quad \text{and} \quad \sum_{n=0}^N \left\| |x-n|^{-1} \right\|_{L^{1,\infty}} \approx N.
\]
Proof. Because the quasi-norm is positively homogeneous, we need only verify (6) in the case that $f$ and $g$ have quasi-norm comparable to one. We may also assume that $f = \sum 2^n \chi_{F_n}$ and $g = \sum 2^m \chi_{E_m}$. By the normalization just mentioned,

$$\sum_n (2^n |F_n|^{1/p})^q \approx 1 \approx \sum_m (2^m |E_m|^{1/p'})^q$$

Combining the above with Lemma 4, we obtain

$$\sum_{A \in \mathcal{E}} \left( \sum_{n:F_n \cap A} 2^n A^{1/p} \right)^q \approx \sum_{A \in \mathcal{E}} \sum_{n:F_n \cap A} (2^n |F_n|^{1/p})^q \approx 1.$$  

and similarly for $g$. Now we compute:

$$\int |fg| \, dx = \sum_{n,m} 2^n 2^m |F_n \cap E_m|$$

$$\leq \sum_{A,B \in \mathcal{E}} \left( \sum_{n:F_n \cap A} 2^n \right) \cdot \min(A, B) \cdot \left( \sum_{m:E_m \cap B} 2^m \right)$$

$$\leq \sum_{A,B \in \mathcal{E}} \left( \sum_{n:F_n \cap A} 2^n A^{1/p} \right) \cdot \min\left( \left[ \frac{1}{p} \right], \left[ \frac{1}{q} \right] \right) \cdot \left( \sum_{m:E_m \cap B} 2^m B^{1/p'} \right).$$

Notice that this has the structure of a bilinear form: two vectors (indexed over $2^\mathbb{Z}$) with a matrix sitting between them. Moreover, by Schur’s test, the matrix is a bounded operator on $\ell^2(2^\mathbb{Z})$. Thus,

$$\int |fg| \, dx \lesssim \left\| \sum_{n:F_n \cap A} 2^n A^{1/p} \right\|_{\ell^2(\mathbb{Z})} \cdot \left\| \sum_{m:E_m \cap B} 2^m B^{1/p'} \right\|_{\ell^2(\mathbb{Z})} \approx 1$$

by (8) and the corresponding statement for $g$. This completes proof of the $\lesssim$ part of (6). We turn now to the opposite inequality. Given $f = \sum 2^n \chi_{F_n} \in L^{p,q}$, we choose

$$g = \sum_n \left( 2^n |F_n|^{1/p} \right)^{q-1} |F_n|^{-1/p} \chi_{F_n} = \sum_n 2^n (q-1) |F_n|^{q/p} \chi_{F_n}.$$  

Then

$$\int fg = \sum_n \left( 2^n |F_n|^{1/p} \right)^{q-1} 2^n |F_n|^{1-1/p} = \sum_n \left( 2^n |F_n|^{1/p} \right)^q \approx \|f\|_{L^{p,q}} \approx 1.$$  

It remains to show that $\|g\|_{L^{p',q'}} \lesssim 1$. By Proposition 3,

$$\left(\left\|g\right\|_{L^{p',q'}}\right)^q \approx \sum_{A \in \mathcal{E}} A^{q'} \sum_{n \in N(A)} \left|F_n\right|^{q'/p'} \approx \sum_{n \in N(A)} 2^n \approx A.$$  

Notice that for each $A$, the sum in $n$ is over part of a geometric series; indeed,

$$n \in N(A) \iff |F_n| \approx A^{-n/p} 2^{-n} \approx A^{-n/q}.$$  

Thus Lemma 4 applies and yields

$$\left(\left\|g\right\|_{L^{p',q'}}\right)^q \approx \sum_{A \in \mathcal{E}} A^{q'} \sum_{n \in N(A)} \left|F_n\right|^{q'/p'} \approx \sum_n 2^n |F_n|^{q/p} \approx 1.$$  

This provides the needed bound on $g$ and so completes the proof of (6).

The fact that LHS(6) is indeed a norm is a purely abstract statement about vector spaces and (separating) linear functionals. The proof that $L^{p,q}$ is complete in this norm differs little from the usual Riesz–Fischer argument.

Let $\ell$ be a continuous linear functional on $L^{p,q}$. By definition, $|\ell(\chi_E)| \lesssim |E|^{1/p}$ and so the measure $E \mapsto \ell(\chi_E)$ is absolutely continuous with respect to Lebesgue measure and so is represented by some locally $L^1$ function $g$. This is the Radon–Nikodym Theorem. By linearity this representation of the functional extends to
simple functions. Boundedness when tested against simple functions suffices to show that \( g \in L^{p,q} \). When \( q \neq \infty \), the simple functions are dense in \( L^{p,q} \) and so our linear functional admits the desired representation.

When \( q = \infty \) the simple functions are not dense. For example, one cannot approximate \( |x|^{-d/p} \in L^{p,\infty}(\mathbb{R}^d) \) by simple functions. Indeed, inspired by the Banach limit linear functionals on \( \ell^\infty(\mathbb{Z}) \) we can construct a non-trivial linear functional on \( L^{p,\infty} \) that vanishes on simple functions. Let \( \mathcal{L} \) denote the vector space of \( f \in L^{p,\infty} \) such that

\[
\ell(f) := \lim_{x \to 0} |x|^{d/p} f(x) \quad \text{exists.}
\]

Notice that \( \mathcal{L} \) contains the simple functions and that \( \ell \) vanishes on these. By the Hahn–Banach theorem, we can extended \( \ell \) to a linear functional on all of \( L^{p,q} \).

**Definition 6.** We say that a mapping \( T \) on (some class of) measurable functions is sublinear if it obeys

\[
|T(cf)(x)| \leq |c||Tf(x)| \quad \text{and} \quad |T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|
\]

for all \( c \in \mathbb{C} \) and measurable functions \( f \) and \( g \) (in the domain of \( T \)).

Linear maps are obviously sublinear. Moreover, if \( \{T_t\} \) is a family of linear maps then

\[
[Tf](x) := \left\| T_tf(x) \right\|_{L_q^q}
\]

is sublinear. The case \( q = \infty \) yields a kind of ‘maximal function’, while \( q = 2 \) gives a kind of ‘square function’.

**Theorem 7** (Marcinkiewicz interpolation theorem). Fix \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \) with \( p_0 \neq p_1 \) and \( q_0 \neq q_1 \). Let \( T \) be a sublinear operator that obeys

(9) \[
\int |\chi_E(x) |T\chi_J(x)| dx \lesssim |E|^{1/q_j} |F|^{1/p_j} \quad j \in \{0,1\}
\]

uniformly for finite-measure sets \( E \) and \( F \). Then for any \( 1 \leq r \leq \infty \) and \( \theta \in (0,1) \),

\[
\|Tf\|_{L^\theta,r} \lesssim \|f\|_{L^{p_0,r}}
\]

where \( 1/p_0 = (1-\theta)/p_0 + \theta/p_1 \) and similarly, \( q_0 = (1-\theta)/q_0 + \theta/q_1 \).

**Remarks.**

1. This form of the result is actually due to Hunt. The original version is Corollary 8 below.

2. Inequalities of the form (9) are known as restricted weak type estimates. Note

\[
\int |\chi_E |T\chi_F| dx \lesssim |E|^{1/q'} |F|^{1/p} \iff \|T\chi_F\|_{L^\infty} \lesssim |F|^{1/p} \iff \|Tf\|_{L^{\infty}} \lesssim \|f\|_{L^p}
\]

as can be shown using Propositions 3 and 5. The rightmost inequality here is called a weak type estimate. At the top of the food chain sits the strong type estimate: \( \|Tf\|_{L^p} \lesssim \|f\|_{L^p} \). If \( p_0 \leq q_0 \) we then can choose \( r = q_0 \) and so (using the nesting of Lorentz spaces) obtain a strong type estimate as the conclusion of the theorem.

3. The hypothesis \( p_0 \leq q_0 \) is needed to obtain the strong type conclusion. Consider, for example,

\[
f(x) \mapsto x^{-1/2} f(x) \quad \text{which maps} \quad L^p([0,\infty), dx) \to L^{2p/(p+2),\infty}([0,\infty), dx)
\]

boundedly for all \( 2 \leq p \leq \infty \). However

\[
f(x) = x^{-1/p} \log(x + x^{-1})\left[\frac{1}{4} \left( 1 - \frac{1}{2} \right)^{-1/2} \right]
\]

shows that \( T \) does not map \( L^p \) into \( L^{2p/(p+2)} \) for any such \( p \).
Proof of Theorem 7. By the duality relations among Lorentz spaces (cf. Proposition 5), it suffices to show that
\[ \left| \int g(x)[Tf](x) \, dx \right| \lesssim 1 \quad \text{whenever} \quad \|f\|_{L^{p_0, r}} \approx 1 \approx \|g\|_{L^{q_0', r'}}. \]

Moreover, we can take \( g \) to be of the form \( \sum 2^n \chi_{E_n} \).

We would like to take \( f \) of the same form, but this takes a little more justification. First by splitting a general \( f \) into real/imaginary parts and then each of these into its positive/negative parts, we see that it suffices to consider non-negative functions \( f \). This also justifies taking \( g \) of the special form. Note that for \( g \) we can safely round up to the nearest power of two; however, since \( T \) need not have any monotonicity properties we are not able to do this for \( f \).

Now by using the binary expansion of the values of \( f(x) \geq 0 \) at each point, we see that it is possible to write \( f \) as the sum of a sequence functions of the form \( \sum 2^n \chi_{F_n} \) in such a way the summands are bounded pointwise by \( f, \frac{1}{2} f, \frac{1}{4} f \), and so on. Since \( L^{p_0, q_0} \) is a Banach space (specifically the triangle inequality holds) we can just sum the pieces back together. (A similar decomposition is possible under a quasi-norm, but a little cunning is required to avoid the summability being swamped by the constants from the triangle inequality.)

Now we have reduced to considering \( f = \sum 2^n \chi_{F_n} \) and \( g = \sum 2^n \chi_{E_n} \), let us compute:
\[ \int |g(x)[Tf](x)| \, dx \lesssim \sum_{n,m} 2^n 2^m \min_{j \in \{0,1\}} \left( |F_n|^{1/p_0} |E_m|^{1/q_0'} \right) \]
\[ \lesssim \sum_{A,B \in 2^\mathbb{Z}} \left( \sum_{n:|F_n| \sim A} 2^n A^{1/p_0} \right) \min_{j \in \{0,1\}} \left( A^{\frac{1}{p_0}} B^{\frac{1}{q_0'}} \right) \left( \sum_{m:|E_m| \sim B} 2^m B^{1/q_0'} \right). \]

Once again we recognize the structure of a bilinear form with vectors indexed over \( 2^\mathbb{Z} \). With a little effort, we see that the matrix has the form
\[ \min_{j \in \{0,1\}} \left( A^{\frac{1}{p_0}} B^{\frac{1}{q_0'}} \right)^{\frac{1}{2}} \]
and so is bounded on \( \ell^r(2^\mathbb{Z}) \) by Schur’s test. (It is essential here that \( p_0 \neq p_1 \) and \( q_0 \neq q_1 \).) On the other hand, by Lemma 4,
\[ \sum_{A \in 2^\mathbb{Z}} \left( \sum_{n:|F_n| \sim A} 2^n A^{1/p_0} \right)^r \approx \sum_n (2^n |F_n|^{1/p_0})^r \approx \left( \|f\|_{L^{p_0, r}} \right)^r \approx 1 \]
and similarly for \( g \), though we use power \( r' \). Putting these all together completes the proof. \( \square \)

Corollary 8 (Marcinkiewicz interpolation theorem). Suppose \( 1 \leq p_0 < p_1 \leq \infty \) and \( T \) is a sublinear operator that obeys
\[ \|Tf\|_{L^{p_0, \infty}} \lesssim \|f\|_{L^{p_0}} \quad \text{and} \quad \|Tf\|_{L^{p_1, \infty}} \lesssim \|f\|_{L^{p_1}} \]
uniformly for measurable functions \( f \). Then for any \( \theta \in (0,1) \),
\[ \|Tf\|_{L^{p_0}} \approx \|f\|_{L^{p_0}} \]
where \( 1/p_0 = (1-\theta)/p_0 + \theta/p_1 \) and similarly, \( q_0 = (1-\theta)/q_0 + \theta/q_1 \).