247B Homework.

1. Suppose \( g = g^* \) is strictly radial decreasing. Show that
\[
\int_{\mathbb{R}^d} f(x)g(x) \, dx = \int_{\mathbb{R}^d} f^*(x)g^*(x) \, dx
\]
implies \( f = f^* \) almost everywhere. (We discussed this all too briefly in class; I’m asking you to fill in the details.)

2. Read sections I and II of G. H. Hardy and J. E. Littlewood, [A maximal theorem with function-theoretic applications](Acta Math. 54, 81–116 (1930)]. This explains the relation between cricket, rearrangement inequalities, and the maximal theorem.

3. Fix \( 1 \leq p < \infty \) and let \( f_n \) be a sequence in \( L^p \) with \( \|f_n\| = 1 \). Suppose \( f_n \to \phi \) in \( L^p \) on any compact set (for some \( \phi \in L^p \)) and write \( f_n = \phi + r_n \). Show that
\[
\|\phi\|_{L^p}^p + \|r_n\|_{L^p}^p \to 1
\]
as \( n \to \infty \).

4. Let \( u \in S(\mathbb{R}^3) \) be a real valued solution to \( -\Delta u - u^5 = -u \); show that \( u \equiv 0 \).

**Remarks:**
(i) This is a special case of Pohožaev’s Non-existence Theorem, which says that the same is true in other dimensions \( d \geq 3 \) for the equation \( \Delta u + (|u|^{4/d-2} - 1)u = 0 \) as well as in star-shaped domains with Dirichlet boundary conditions. The proof of the more general result is essentially the same as for this special case.

(ii) Standard results in the theory of elliptic PDE can be used to show that any \( H^1(\mathbb{R}^3) \) solution to our equation must actually be Schwartz.

(iii) The power five corresponds to the variational problem for Sobolev embedding; for smaller powers, this equation does have non-trivial solutions, namely the Gagliardo–Nirenberg optimizers.

(iv) The key idea is to integrate the equation against \( \bar{u} \) and \( x \cdot \nabla \bar{u} \) to obtain inconsistent identities. The inspiration for the second multiplier is undoubtedly the role of dilations in this problem (cf. below). In particular, attempting to find solutions to the PDE as optimizers of natural functional will not work work since rescaling any putative optimizer produces a better answer.

5. For \( f \in S(\mathbb{R}^d) \) solve
\[
\partial_t u(t, x) = x \cdot \nabla u(t, x) \quad \text{with} \quad u(0, x) = f(x).
\]
by observing this is precisely the vanishing of a directional derivative in space-time. (This is called the ‘method of characteristics’ and the space-time curves along which \( u \) is forced to be constant are called the ‘characteristics’.) Now solve
\[
\partial_t u(t, x) = \frac{1}{2} \left[ \nabla \cdot (xu(t, x)) + x \cdot \nabla u(t, x) \right] \quad \text{with} \quad u(0, x) = f(x).
\]
Note that the RHS is (formally at least) an anti-self-adjoint operator on \( L^2(\mathbb{R}^d) \), so we expect to find a one-parameter group of unitary mappings.