247A Homework.

1. Show that the dual of the LCA group $\mathbb{R}$ is isomorphic to $\mathbb{R}$.
2. Let G be a finite abelian group and H a subgroup. For $\chi \in \hat{G}$ we write
$$\hat{f}(\chi) = \sum_g f(g)\chi(g).$$
We say $\chi \in \hat{G}^H$ if χ is constant on each coset of H. Prove
(a) $\hat{G}^H \cong \widehat{G/H}$ in a natural way; and
(b) the following 'Poisson Summation' formula:
$$\frac{|H|}{|G|} \sum_{\chi \in \hat{G}^H} \hat{f}(\chi) = \sum_{h \in H} f(h).$$
3. Recall from class that the dyadic cubes in $\mathbb{R}^d$ are the sets of the form
$$Q_{n,k} = [k_1 2^n, (k_1+1)2^n) \times \cdots \times [k_d 2^n, (k_d+1)2^n)$$
were n ranges over $\mathbb{Z}$ and $k \in \mathbb{Z}^d$. We define $\mathcal{F}_n$ as the smallest σ-algebra containing every $Q_{-n,k}$.
(a) Given a collection of dyadic cubes whose diameters are bounded, show that one may find a sub-collection which covers the same region of $\mathbb{R}^d$ but with all cubes disjoint.
(b) Define the (uncentered) dyadic maximal function by
$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y)\, dy$$
where the supremum is over all dyadic cubes that contain x. Show that this operator is of weak type (1,1).
(c) Fix $1 \le p \le \infty$. Show that for any $f \in L^p(\mathbb{R}^d)$,
$$\mathbb{E}(f|\mathcal{F}_n) \to f$$
both almost everywhere and (when $1 \le p < \infty$) in L^p sense.
4. Suppose $f \in L^1(\mathbb{R}/\mathbb{Z})$ and let $M_D f$ denote its (uncentred) dyadic maximal function.
(a) Show that for $\lambda > \int |f|$,
$$\frac{1}{\lambda} \int_{|f|>\lambda} |f(x)|\, dx \lesssim |\{x : M_D f > \lambda\}|.$$
[Hint: Do a Calderón–Zygmund style decomposition.]
(b) Deduce that if $M_D f \in L^1$, then $|f| \log[1 + |f|] \in L^1$. This result is due to Stein.
(c) Use the fact that $M_D : L^\infty \to L^\infty$ and $L^1 \to L^{1,\infty}$ to show
$$|\{x : Mf > \lambda\}| \lesssim \frac{1}{\lambda} \int_{|f|>c\lambda} |f(x)|\, dx.$$
for some small constant c.
(d) Deduce that if $|f| \log[1 + |f|] \in L^1$ then $Mf \in L^1$.
5. Given a Schwartz vector field $F : \mathbb{R}^3 \to \mathbb{C}^3$, define vector and scalar fields A and ϕ via
$$\hat{\phi}(\xi) = \frac{\xi \cdot \hat{F}(\xi)}{2\pi i |\xi|^2} \quad \text{and} \quad \hat{A}(\xi) = -\frac{\xi \times \hat{F}(\xi)}{2\pi i |\xi|^2}.$$

Note that ϕ and A are smooth functions, but need not be Schwartz.
(a) Show that

$$\|\phi\|_{L^q(\mathbb{R}^3)} + \|A\|_{L^q(\mathbb{R}^3)} \lesssim \|F\|_{L^p(\mathbb{R}^3)}$$

for $1 < p < q < \infty$ obeying $1 + \frac{d}{q} = \frac{d}{p}$.
(b) Show that $F = \nabla \times A + \nabla\phi$ and hence that

$$\|F\|_{L^p(\mathbb{R}^3)} \approx \|\nabla \times A\|_{L^p(\mathbb{R}^3)} + \|\nabla\phi\|_{L^p(\mathbb{R}^3)}$$

for any $1 < p < \infty$.
(c) Show that all (first-order) derivatives of all components of A are under control (not just the curl):

$$\|\partial_k A_l\|_{L^p(\mathbb{R}^3)} \lesssim \|F\|_{L^p(\mathbb{R}^3)}$$

for any $1 < p < \infty$ and any $k, l \in \{1, 2, 3\}$.

Remark: Observe that $F = \nabla \times A + \nabla\phi$ decomposes F into a divergence-free part and a curl-free part. Indeed this (Helmholtz–Hodge) decomposition is orthogonal under the natural inner-product on vector-valued functions. Note however, that the choice of A is far from unique; consider $A \mapsto A + \nabla\psi$. Our choice corresponds to the Coulomb gauge: $\nabla \cdot A = 0$.

The end