1. Let Ω be a simply-connected open domain bounded by a Jordan curve. As we know, any conformal map f of $\mathbb{D}$ onto Ω can be extended to a homeomorphism of $\bar{\mathbb{D}}$ onto $\bar{\Omega}$.

Recall that a curve $\gamma : \partial\mathbb{D} \to \mathbb{C}$ is rectifiable if there exists a constant L so that for any $0 \leq \theta_0 < \theta_1 < \cdots < \theta_n \leq 2\pi$,

$$\sum_{k=0}^{n} |\gamma(e^{i\theta_k}) - \gamma(e^{i\theta_{k+1}})| \leq L$$

where $\theta_{n+1} = \theta_0$. The minimal such constant L is called the *length* of γ.

Prove the following theorem of F. and M. Riesz: $f' \in H^1$ if and only if $\partial\Omega$ is rectifiable. [*Hint*: the function $z \mapsto \sum |f(ze^{i\theta_k}) - f(ze^{i\theta_{k+1}})|$ is continuous and subharmonic on $\mathbb{D}$.]

2. Continuing from the previous problem, let $f'(e^{i\theta})$ denote the a.e. boundary values of $f'(z)$ and let ϕ denote the homeomorphism of $\partial\mathbb{D}$ onto $\partial\Omega$ induced by f.
(a) Show that ϕ is absolutely continuous with derivative $ie^{i\theta} f'(e^{i\theta})$.
(b) Deduce that

$$\text{Length}(\phi(E)) = \int_E |f'(e^{i\theta})|\, d\theta$$

when E is a closed arc in $\partial\mathbb{D}$. By abstract measure theory, this induces a positive measure (arc length measure) on all Borel subsets of $\partial\Omega$.
(c) Show that the inverse homeomorphism $\phi^{-1} : \partial\Omega \to \partial\mathbb{D}$ is absolutely continuous in the sense that it takes sets of zero arc length to sets of zero Lebesgue measure.

3. Modify the proof that the Hardy-Littlewood maximal function is bounded on L^p for $1 < p < \infty$ to show that

$$\int |f(e^{i\theta})| \log\big[2 + |f(e^{i\theta})|\big]\, d\theta < \infty \implies \int |[Mf](e^{i\theta})|\, d\theta < \infty.$$

4. Prove (following Euler) that the number of partitions of n into *distinct* parts is equal to the number of partitions of n into *odd* parts by writing the generating function for each as an infinite product and then comparing.

5. Recall the beta integral from last quarter (cf. HW1 Problem 8):

$$B(a+1, b+1) = \int_0^1 t^a (1-t)^b\, dt$$

with a close relationship to the binomial coefficients. Determine the leading-term asymptotics of this integral for $a = pn$, $b = (1-p)n$ with $n \to \infty$ and $p \in (0,1)$ fixed. Here, *leading-term* means such that the relative error converges to zero.