1. Let $\Omega_r := \{z \in \mathbb{D} : |z| > r\}$ so Ω_0 denotes the punctured disk.
(a) Solve the Dirichlet problem for Ω_r with $0 < r < 1$ for the following boundary data: $\phi(z) = 0$ if $|z| = 1$ and $\phi(z) = 1$ if $|z| = r$. Note the behaviour as $r \downarrow 0$.
(b) Use this and the maximum principle to show that if $u : \Omega_0 :\to \mathbb{R}$ is bounded and harmonic then it admits a continuous extension to all of $\mathbb{D}$ and this extension is harmonic.

2. (a) Find a bounded harmonic function u on $\mathbb{C}^+ := \{z : \operatorname{Im} z > 0\}$ so that

$$\lim_{y \downarrow 0} u(x+iy) = \begin{cases} 0 & : x > 0 \\ 1 & : x < 0 \end{cases}$$

for all $x \in \mathbb{R} \setminus \{0\}$.
(b) Fix $\alpha \in (0, 2\pi)$. Use this and Problem 1(d) from HW5 to find a harmonic function $u : \Omega'_\alpha \to [0, 1]$ which vanishes on (the interior of) the straight edges of the boundary and is equal to one on (the interior of) the arc.
(c) Use this to formulate and prove a sharp form of Phragmén–Lindelöf Principle for the sector $\Omega_\alpha := \{re^{i\theta} : 0 < r < \infty \text{ and } 0 < \theta < \alpha\}$.

3. Assuming Stirling's formula:

$$\log[\Gamma(z+1)] = z\log(z) - z + \tfrac{1}{2}\log(2\pi z) + O\big(|z|^{-1}\big)$$

uniformly as $|z| \to \infty$ in any sector $|\arg(z)| < \pi - \delta$. Use the Phragmén–Lindelöf principle to obtain a bound for the growth of $\zeta(\frac{1}{2}+it)$ by a fractional power of $(1+|t|)$. [*Remark:* The (unresolved) Lindelöf Hypothesis is the claim that the truth is $O(t^\epsilon)$ for any $\epsilon > 0$. The first result to beat the technique suggested here is $O(t^{\frac{1}{6}+\epsilon})$ due to Weyl.]

4.

5.

6.