1. Give a direct proof of Montel's Theorem from the Cauchy Integral and Arzelà–Ascoli Theorems.

2. Prove *Hurwitz Theorem:* Suppose $\{f_n\}$ and f are holomorphic functions on an open set $\Omega \subset \mathbb{C}$. If $f_n \to f$ uniformly on compact sets and $f \not\equiv 0$, then the zeros of f_n converge to those of f in the following sense: given a ball B with $\bar{B} \subseteq \Omega$ and f nowhere zero on ∂B, the number of zeros (counting multiplicity) of f_n in B converges to the number for f. [*Remark:* Since the number is an integer, convergence means eventual equality.]

3. (a) Suppose $\Omega \subset \mathbb{C}$ is open and connected. Let $f_n : \Omega \to \mathbb{C}$ be univalent (=holomorphic and injective) and converge uniformly on compact sets to some (holomorphic) $f : \Omega \to \mathbb{C}$. Show that f is univalent or constant.

(b) Suppose Ω has compact closure and $z_0 \in \Omega$. Show that among all univalent maps $f : \Omega \to \mathbb{D}$ that obey $f(z_0) = 0$, there is (at least) one that achieves the maximal value of $\operatorname{Re} f'(z_0)$. [*Note:* you will need to verify that the set of maps is non-empty and that $\operatorname{Re} f'(z_0)$ is bounded.]

(c) Note that any such maximal f has $f'(z_0) > 0$.

4. Now suppose that Ω is simply connected and let $f : \Omega \to \mathbb{D}$ be one of the univalent maps found in Problem 3(b). To obtain a proof of the Riemann Mapping Theorem, we need only show that f is onto. Suppose not and let $w_0 \in \mathbb{D}$ be a point not in the range of f.

(a) Show that there is a univalent function $r : \Omega \to \mathbb{D}$ so that

$$r(z)^2 = M_1 \circ f(z)$$

where M_1 is a disk automorphism taking w_0 to 0.

(b) Choose a disk automorphism M_2 so that

$$g(z) = M_2 \circ r(z)$$

obeys $g(z_0) = 0$ and $g'(z_0) \geq 0$.

(c) Rearrange the above to find $\phi : \mathbb{D} \to \mathbb{D}$ so that $f(z) = \phi \circ g(z)$.

5. Continuing from the preceding problems:

(a) Use Schwarz Lemma to see that $|\phi'(0)| < 1$.

(b) Now get your hands dirty and actually compute $\phi'(0)$.

(c) Conclude that any maximizing f from Problem 3 must be onto.

(d) Show that f is actually unique.

6. Let $\Omega \subset \mathbb{C}$ be open and let $\{f_n\}$ be a sequence of meromorphic functions on Ω. Let us write F_n for the corresponding maps from $\Omega \subset \mathbb{R}^2$ to the unit sphere in $\mathbb{R}^3$ induced via stereographic projection.

(a) Use the Cauchy–Riemann equations to show that

$$\|\nabla F_n(x,y)\|^2 = 8\frac{|f_n'(x+iy)|^2}{(1+|f_n(x+iy)|^2)^2}$$

(at poles, the RHS should be regarded as its limiting value).
(b) Prove *Marty's Theorem:* If the above is uniformly bounded on compact subsets of Ω, then a subsequence of the f_n converge to a meromorphic function. For the purposes of this problem, the constant function $f(z) \equiv \infty$ is granted honourary status as a meromorphic function.