

1. Given  $\alpha \in (0, 2\pi)$ , consider the sector  $\Omega_\alpha := \{re^{i\theta} : 0 < r < \infty \text{ and } 0 < \theta < \alpha\}$ .
- (a) Find a biholomorphism of  $\Omega_\alpha$  to the upper half-plane.
- (b) Find a positive harmonic function  $u$  in  $\Omega_\alpha$  that extends continuously to the boundary and vanishes there. [Note: This shows non-uniqueness for solutions of the Dirichlet Problem in a non-compact setting.]
- (c) Now consider sector  $\Omega'_\alpha := \{z \in \Omega_\alpha : |z| < 1\}$ . Find a positive harmonic function  $u$  in  $\Omega'_\alpha$  that extends continuously to  $\partial\Omega'_\alpha \setminus \{0\}$  and vanishes there. For which values of  $\alpha$  is this function square integrable in the sector?
- (d) Find biholomorphisms from  $\Omega'_\alpha$  to  $\Omega'_\pi$  to  $\Omega_{\pi/2}$  and finally to the upper half-plane.

- 2.(a) Find a positive harmonic function  $u$  on the strip  $\{z : 0 < \text{Im } z < 1\}$  that extends continuously to the boundary and vanishes there.
- (b) Find a biholomorphism of this strip and the upper half-plane.
- (c) Find a positive harmonic function on the half-strip  $\{z : 0 < \text{Im } z < 1 \text{ \& } \text{Re } z > 0\}$  that extends continuously to the boundary and vanishes there.
- (d) Find a biholomorphism from this half-strip to the upper half-plane.

- 3.(a) Show that

$$\int [\Delta\phi](x + iy) \frac{1}{2\pi} \log|x + iy| dx dy = \phi(0)$$

for all  $\phi \in C_c^\infty(\mathbb{C})$ .

- (b) Let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic, but not identically zero. Show that  $\log|f(z)|$  is sub-harmonic in the sense of distributions. (Even though  $\log|f(z)|$  need not be continuous, it is Riemann integrable, so the question makes sense.)

4. Let  $\Omega$  be a open set in  $\mathbb{C}$  bounded by a  $C^\infty$  Jordan curve. We recall that  $G_p : \Omega \setminus \{p\} \rightarrow \mathbb{R}$  is the *Dirichlet Green function* for  $\Omega$  with a pole at  $p \in \Omega$  if  $G$  is harmonic,

$$\lim_{z \rightarrow p} (G_p(z) - \frac{1}{2\pi} \log|z - p|) \quad \text{exists,}$$

and  $G$  extends continuously to  $\partial\Omega$  and vanishes there. Kellogg's Theorem tells us that  $G$  has a  $C^\infty$  extension to a neighbourhood of the boundary. Suppose  $u$  is  $C^\infty$  in a neighbourhood of  $\bar{\Omega}$  and is harmonic in  $\Omega$ , show that

$$u(p) = \int_{\partial\Omega} \frac{dG}{d\nu}(y) u(y) dS(y)$$

where  $\frac{d}{d\nu}$  denotes the outward normal derivative. [Remark: The regularity hypotheses are far stronger than are really necessary.]

5. Find the Green function and Poisson kernel for
- (a) The upper half-plane.
- (b) The sector  $\Omega_\alpha$  from Problem 1.
- (c) The strip  $0 < \text{Re } z < 1$ .

We require that  $G_p(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ , which breaks the non-uniqueness problem concomitant with the case of non-compact domains. The Poisson kernel is then defined via normal derivatives as above.

6. Use the Poisson integral formula to prove the following version of Harnack's inequality for the unit ball  $B(0, 1) \subseteq \mathbb{R}^n$  (and  $n \geq 2$ ): If  $u : B(0, 1) \rightarrow [0, \infty)$  is harmonic, then

$$\frac{1 - |x|}{(1 + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{1 + |x|}{(1 - |x|)^{n-1}} u(0)$$

for each  $|x| < 1$ . Observe that equality can occur, so these estimates are best possible.

7.(a) Suppose  $f \in C_c^\infty(\mathbb{R}^3)$ . Show that

$$-\Delta \int_{\mathbb{R}^3} \frac{f(y)}{4\pi|x-y|} dy = f(x)$$

(b) [Newton's Theorem] Suppose  $f$  is radially symmetric. Show that

$$\int_{\mathbb{R}^3} \frac{f(y)}{4\pi|x-y|} dy = \frac{1}{4\pi|x|} \int_{\mathbb{R}^3} f(y) dy$$

whenever  $|x|$  is larger than  $\sup\{|y| : f(y) \neq 0\}$ .

[*Remark:* This says that (outside the body) the gravitational field of a spherical body is indistinguishable from that due to a point particle of equal mass. Interestingly, this theorem is also true in General Relativity. What would happen inside a hollow spherical planet?]