

1. (a) Show that every Möbius transformation f conforms to exactly one of the following adjectives:

Identity: It is the identity map.

Hyperbolic: It is conjugate to $z \mapsto \lambda z$ for some $\lambda > 1$.

Parabolic: It is conjugate to $z \mapsto z + 1$.

Elliptic: It is conjugate to $z \mapsto e^{i\theta} z$.

Loxodromic: It is conjugate to $z \mapsto \lambda z$, with $|\lambda| > 1$ and $\text{Im } \lambda > 0$.

Include a simple diagnostic in terms of the coefficients of the Möbius transformation.

Here ‘conjugate’ means in group of Möbius transformations.

(b) Describe the asymptotic behaviour of $z \in \mathbb{C} \cup \{\infty\}$ under iteration of f .

(c) Show that for automorphisms of \mathbb{D} (or $\mathbb{H} = \{z : \text{Im } z > 0\}$), the loxodromic case does not occur.

2. Define the *hyperbolic (or Poincaré) distance* between two $z_1, z_2 \in \mathbb{D}$ as

$$d(z_1, z_2) = \log \left[\frac{1 + \rho}{1 - \rho} \right] \quad \text{with} \quad \rho := \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|$$

(a) Show that this is indeed a metric on \mathbb{D} and that

$$d(w_1, w_2) = d(z_1, z_2) \quad \text{whenever} \quad w_j = g(z_j)$$

with g a (Möbius) automorphism of the unit disk.

(b) In fact, there is a disk automorphism mapping $z_1, z_2 \in \mathbb{D}$ to $w_1, w_2 \in \mathbb{D}$ if and only if $d(w_1, w_2) = d(z_1, z_2)$. Show this too.

(c) Show that with the exception of the disk automorphisms, any holomorphic map $f : \mathbb{D} \rightarrow \mathbb{D}$ is a contraction in the hyperbolic metric, that is, $d(f(z_1), f(z_2)) < d(z_1, z_2)$.

[*Hint:* you may find it most efficient to solve the parts out of order.]

3. The *Schur algorithm* is a recursive procedure takes as input a holomorphic map $f_0 : \mathbb{D} \rightarrow \bar{\mathbb{D}}$ (here $\bar{\mathbb{D}}$ denotes the closed unit disk) and returns a finite or infinite sequence of complex numbers:

Set $k = 0$ and define

$$(*) \quad \alpha_k := f_k(0) \quad \text{and} \quad f_{k+1}(z) := \frac{1}{z} \cdot \frac{f_k(z) - \alpha_k}{1 - \bar{\alpha}_k f_k(z)}.$$

Output α_k . If $|\alpha_k| = 1$ we stop; otherwise, we increment k and return to (*).

(a) Verify that $f_{k+1} : \mathbb{D} \rightarrow \bar{\mathbb{D}}$.

(b) Show (perhaps recursively) that $\alpha_0, \dots, \alpha_m$ determine $f(0), f'(0), \dots, f^{(m)}(0)$ for each $m \geq 0$, and conversely. If there are fewer than $m + 1$ coefficients (because the algorithm terminated) this smaller subset still suffices.

(c) Deduce that the mapping from f_0 to its sequence of coefficients is injective.

(d) Show that any finite sequence

$$\alpha_0, \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1} \in \mathbb{D} \times \mathbb{D} \times \dots \times \mathbb{D} \times \partial\mathbb{D}$$

can occur as the output of our algorithm. We will denote the (unique) input generating this sequence by $f(z; \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$.

(e) Given an infinite sequence $\{\alpha_k \in \mathbb{D}\}_{k=0}^\infty$, show that

$$F_n(z) := f(z; \alpha_0, \dots, \alpha_{n-2}, 1)$$

is uniformly Cauchy on compact subsets of \mathbb{D} . So deduce that there is an input $f : \mathbb{D} \rightarrow \mathbb{D}$ which produces our chosen sequence as output.

[*Remark:* The Schur algorithm was independently co-discovered by Verblunsky and by Herglotz in slightly different settings. The coefficients α_k have at least five different names; today, we will call them *Schur parameters*. Next quarter, we will call them *Verblunsky parameters*, when we discuss the relation to the *circular moment problem*: Given $c_0, \dots, c_m \in \bar{\mathbb{D}}$ is there a positive measure on $[0, 2\pi)$ so that

$$c_k = \int e^{ik\theta} d\mu(\theta).$$

If so, describe all such solutions. This is essentially the problem considered by Verblunsky. Herglotz was interested in holomorphic functions on \mathbb{D} with positive real part.]

4. Show that for each integer $m \geq 1$, the following three sets of mappings have precisely the same elements:

(i) The m -fold holomorphic branched covering maps $f : \mathbb{D} \rightarrow \mathbb{D}$. That is, holomorphic functions f such that for each $w \in \mathbb{D}$ there are exactly m solutions (counting multiplicity) to $f(z) = w$ with $z \in \mathbb{D}$.

(ii) Blaschke products of degree m , that is, functions of the form

$$f(z) = e^{i\phi} \prod_{j=1}^m \frac{z - z_j}{1 - \bar{z}_j z}$$

with $\phi \in [0, 2\pi)$ and $z_1, \dots, z_m \in \mathbb{D}$.

(iii) Holomorphic maps $f : \mathbb{D} \rightarrow \bar{\mathbb{D}}$ with exactly $m + 1$ Schur parameters. Note in particular that $\alpha_m \in \partial\mathbb{D}$.

[*Warning:* I think this problem is the hardest so far; break it up into sub-problems. You may wish to think about using induction on m and/or the Argument Principle.]