1. (a) Use Rouché's Theorem to show that the roots of a polynomial depend continuously on the coefficients of the polynomial. Here, we say that the distance between two multi-sets of roots is the sum of the distances under the shortest matching.
(b) Show by example that the roots of a polynomial of degree n need not be better that Hölder continuous of order $1/n$ as functions of the coefficients.

2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is bijective and holomorphic.
(a) Show that f^{-1} is holomorphic.
(b) Use the continuity of f^{-1} to show that $|f(z)| \to \infty$ as $|z| \to \infty$.
(c) Show that $f(z) = O(|z|)$ as $z \to \infty$. [*Hint:* Look at $g(z) = 1/f(1/z)$.]
(d) Deduce that $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

3. Consider meromorphic functions of the form

$$f(z) = \frac{az + b}{cz + d}.$$

for some quadruplet $a, b, c, d \in \mathbb{C}$ obeying $ad - bc = 1$. These are called, inter alia, *Möbius transformations* or *linear fractional transformations.*
(a) Show that Möbius transformations are bijections on the Riemann Sphere $\mathbb{C} \cup \{\infty\}$. [*Remark:* If we relax the restriction to $ad - bc \neq 0$, we obtain no new functions; if $ad - bc = 0$, then f is no longer a bijection.]
(b) Show that the map

$$\Phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az + b}{cz + d} \right)$$

is a group homomorphism from $\mathrm{SL}(2, \mathbb{C})$, that is, the group of matrices of determinant one, to the group of Möbius transformations (under composition of functions).
(c) Determine the kernel of Φ. [*Remark:* We see that the group of Möbius transformations is therefore isomorphic to the quotient of $\mathrm{SL}(2, \mathbb{C})$ by the kernel of this map, a group known as the projective special linear group, $\mathrm{PSL}(2, \mathbb{C})$.]
(d) Suppose (z_1, z_2, z_3) and (w_1, w_2, w_3) are both triples of distinct points in $\mathbb{C} \cup \{\infty\}$. Show that there is a unique Möbius transformation f that obeys $f(z_j) = w_j$ for each $j \in \{1, 2, 3\}$. [*Hint:* first treat the case $(w_1, w_2, w_3) = (0, 1, \infty)$.]

4. (a) Show that stereographic projection (defined in HW#1) conjugates rigid rotations of the (Riemann) sphere to Möbius transformations and identify the corresponding subgroup of $\mathrm{SL}(2, \mathbb{C})$.
(b) Determine the collection of Möbius transformations that are bijections on the upper half-plane $\{z : \mathrm{Im}\, z > 0\}$. Identify the corresponding subgroup of $\mathrm{SL}(2, \mathbb{C})$.
(c) Hence determine the collection of Möbius transformations that are bijections of the open unit disk.

5. Show that every meromorphic bijection f from $\mathbb{C} \cup \{\infty\}$ to itself is of the form

$$f(z) = \frac{az + b}{cz + d}$$

for some quadruplet $a, b, c, d \in \mathbb{C}$ obeying $ad - bc = 1$.

6.(a) Use Möbius transformations to determine the analogue of the Schwarz Reflection Principle for holomorphic functions f defined in an open neighbourhood of an arc of the circle $\{|z| = 1\}$ that obey $|f(e^{i\theta})| = 1$.
(b) Use Möbius transformations to determine the analogue of Schwarz Lemma for mappings f of the half-plane $\{\operatorname{Re} z > 0\}$ to itself that obey $f(1) = 1$.
(c) Use Möbius transformations and Schwarz Lemma to prove the *Borel–Carathéodory Theorem:* Let f be holomorphic in an open neighbourhood of the closed unit disk, $\bar{\mathbb{D}}$. For each $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|} |f(0)| + \frac{2|z|}{1 - |z|} \sup_{w \in \mathbb{D}} \operatorname{Re} f(w).$$