1. Prove the *Casorati–Weierstrass Theorem:* Suppose f is defined in a deleted neighbourhood of $z_0 \in \mathbb{C}$, say $\{z : 0 < |z - z_0| < r\}$, and is holomorphic there. Suppose also that z_0 is neither a removable singularity nor a pole. Prove that for each $0 < \epsilon < r$ the set $f(\{z : 0 < |z - z_0| < \epsilon\})$ is dense in $\mathbb{C}$.
Hint: Mimic the Lemma from class proving the existence of a principal part at any pole.

2. Let us say that meromorphic function f on $\mathbb{C}$ is *doubly periodic* if there exists $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and

$$f(z+\tau) = f(z+1) = f(z).$$

Notice that we have fixed one of the periods to be 1. This is no real loss of generality since other cases can be reduced to this via a change of variable $z \mapsto az$. The number τ is far from unique as we will discuss later in the course. For now, show that
(a) Holomorphic doubly periodic functions are constant. Indeed,
(b) Nonconstant doubly periodic functions are onto $\mathbb{C} \cup \{\infty\}$, that is, they have a pole and achieve every (finite) value in $\mathbb{C}$.
Remark: It has been reported (though documentary evidence is limited) that Liouville lectured on the result (a), Cauchy objected that this would imply that bounded holomorphic functions are constant, and so what we now call Liouville's Theorem was born. Note that obvious candidate for Cauchy's argument uses the fact that there is a doubly periodic function with property (b), which was known by direct construction (we'll do this later too).

3. Suppose f and g are meromorphic on some open $\Omega \subseteq \mathbb{C}$ and $g \not\equiv 0$. Show that f/g is meromorphic. (Actually all meromorphic functions can be written in this form with f and g holomorphic, but proving this requires technology that we haven't covered yet.)

4. Suppose $\phi : \mathbb{R}^n \to \mathbb{C}$ is C^∞ of compact support and $\int_{\mathbb{R}^n} \phi(y)\, dy = 1$. For each $\lambda > 0$ and continuous function $f : \mathbb{R}^n \to \mathbb{C}$ we define

$$\phi_\lambda(y) = \lambda^{-n}\phi\big(\lambda^{-1}y\big) \quad \text{and} \quad \big[f * \phi_\lambda\big](x) := \int_{\mathbb{R}^n} f(x-y)\phi_\lambda(y)\, dy$$

(a) Show that $\lim_{\lambda\downarrow 0}[f * \phi_\lambda](x) = f(x)$ uniformly for x in compact subsets of $\mathbb{R}^n$.
(b) Observe that $f * \phi_\lambda$ is C^∞.
(c) Now suppose $n = 1$. Show that for each $L > 0$,

$$\text{Length}\big(f * \phi_\lambda\big|_{[-L,L]}\big) \leq \frac{2L}{\lambda} \cdot \int_{\mathbb{R}} |\phi'(y)|\, dy \cdot \sup_{x_1,x_2\in\mathbb{R}} |f(x_1) - f(x_2)|$$

(if the righthand side is infinite, then there's nothing to prove).
(d) Deduce that for general continuous $f : \mathbb{R} \to \mathbb{C}$, the curve $f * \phi_\lambda$ is rectifiable on any compact interval.

5. In class, we saw that

$$\zeta(s) = \frac{1}{2i\cos(\pi s/2)} \int_\gamma \frac{z^{-s}\, dz}{e^{-2\pi z} - 1}.$$

Initially this was for $\operatorname{Re} s > 1$ and $s \notin 2\mathbb{Z}+1$ for other values of s the integral can be regarded as the definition of $\zeta(s)$ (as a meromorphic function). By using results from the previous homework and

$$\frac{1}{e^{-2\pi z}-1} = \frac{e^{2\pi N z}}{e^{-2\pi z}-1} + \sum_{n=1}^{N} e^{2\pi n z}$$

show that ζ obeys the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Deduce that $s = 1$ is the only (non-removable) singularity of ζ.
Warnings/Hints: Work first with $\operatorname{Re} s < 0$ then use the uniqueness theorem — what makes this situation different from $\log(z)$ where there are multiple continuations? Be particularly careful with the part of the contour(s) sitting in the right half-plane; here $e^{2\pi N z}$ can be enormous! Move the contour!

6. Evaluate

$$\int_{\mathbb{R}} e^{-t^2/2} e^{i\xi t}\, dt$$

for all $\xi \in \mathbb{C}$. Deduce *Wick's Theorem* (as it is called in the physics literature):

$$\tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^{2k} e^{-t^2/2}\, dt = (2k-1)!! = (2k-1)(2k-3)\cdots(3)(1).$$

Convince yourself that this is the number of ways of pairing off $2k$ objects.