1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be holomorphic. Suppose that for each compact set $K \subset \Omega$ the functions f_n converge uniformly to some $f : \Omega \to \mathbb{C}$. Show that f must me holomorphic. Further, show that $f_n'(z) \to f'(z)$ uniformly on compact subsets of Ω.

2. (a) Prove *Liouville's Theorem*: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$|f(z)| \leq C(1 + |z|)^n$$

for some $C > 0$ and integer $n \geq 0$, then f is a polynomial of degree not exceeding n.
(b) Let Ω be an open neighbourhood of $0 \in \mathbb{C}$. Suppose g is holomorphic on $\Omega \setminus \{0\}$ and obeys

$$|g(z)| \leq C|z|^{-n}$$

there (with n and C as before). Show that there is a holomorphic function $h : \Omega \to \mathbb{C}$ and coefficients $a_1, \ldots, a_n$ in $\mathbb{C}$ so that

$$g(z) = h(z) + \sum_{k=1}^{n} a_k z^{-k}.$$

The sum here is called the *principal part* of g at the point 0.
Hint: First treat the case $n = 0$, for which the sum is empty. This result is known as *Riemann's removable singularity theorem*. For general n consider $f(z) = z^n g(z)$.

3. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, let us define

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

(as an improper Riemann integral).
(a) Prove that Γ is holomorphic on this region.
(b) Show that $z\Gamma(z) = \Gamma(z + 1)$ when $\mathrm{Re}(z) > 0$.
(c) Deduce that $\Gamma(n + 1) = n!$ when $n \geq 0$ is an integer.
(d) Argue that there is a (unique) extension of Γ to a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ that obeys $z\Gamma(z) = \Gamma(z+1)$. Show that the omitted points are polar singularities and determine their principal part.

4. For $j \in \{1, 2\}$, let $\gamma_j : (-1, 1) \to \mathbb{R}^n$ be C^1 curves. Suppose also that $\gamma_1(0) = \gamma_2(0) =: z_0$, and that $\gamma_j'(0) \neq 0$ for both curves. We define the angle between the two curves as that $\theta \in [0, \pi]$ so that

$$\frac{\langle \gamma_1'(0), \gamma_2'(0) \rangle_{\mathbb{R}^n}}{\|\gamma_1'(0)\| \, \|\gamma_2'(0)\|} = \cos\theta.$$

We extend this notion to $\mathbb{C}$ via the usual identification with $\mathbb{R}^2$.
(a) Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at the point z_0 and $f'(z_0) \neq 0$. Show that the curves $f \circ \gamma_j$ subtend the same angle as the original curves γ_j.
(b) Show that the stereographic projection is also a conformal map. You may prefer to verify this for the inverse map,

$$(x, y) \in \mathbb{R}^2 \mapsto \tfrac{1}{1+x^2+y^2}\big(2x, 2y, x^2 + y^2 - 1\big) \in \mathbb{R}^3,$$

which is, of course, equivalent.

5. (a) Show that

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

defines a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.
(b) Show that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

(you may take the fundamental theorem of arithmetic for granted), but you must address the issue of convergence.
(c) Identify a function $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ so that

$$f(n) = \int_{-1/2}^{1/2} f(n+x)\,dx + \int_{-1/2}^{1/2} f''(n+x)g(x)\,dx$$

for all C^∞ functions f defined in a neighbourhood of $[-\frac{1}{2}, \frac{1}{2}]$.
[Remark: one may view this formula as giving the error made when using the midpoint rule of numerical integration.]
(d) Use part (c) to show that we can extend the definition of ζ to make it meromorphic in the region $\mathrm{Re}(s) > -1$. Identify the (single) pole and its residue.

6. Use the residue theorem to evaluate

$$\int_{\mathbb{R}} \frac{e^{i\xi x}\,dx}{1+x^2}$$

for all values of $\xi \in \mathbb{R}$.

7. For arbitrary $s \in \mathbb{C}$, we define

$$f(s) = \frac{1}{2\pi i} \int_\gamma z^{-s} e^z\,dz$$

where z^{-s} takes its principal branch and γ is the contour $\gamma(t) = \frac{1}{2} - |t| + it$ for $t \in \mathbb{R}$.
(a) Show that f is an entire function.
(b) By collapsing the contour onto the negative real axis, show that

$$f(s) = \tfrac{1}{\pi} \sin(\pi s)\Gamma(1-s),$$

at least when $\mathrm{Re}\, s < 1$.

8. (a) By changing variables to $s = uv$ and $t = u(1-v)$ in

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-s-t}\,ds\,dt$$

deduce *Euler's Beta Integral:*

$$\int_0^1 v^{x-1}(1-v)^{y-1}\,dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

(b) Now deduce *Euler's Reflection Formula:*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

Hint: Set $y = 1 - x$ and $e^u = v/(1-v)$. To evaluate the u integral, try moving the contour up by $2\pi i$.