170B Killip | Midterm 2 | May 24th

First Name: ____________________ ID#____________________

Last Name: ____________________

Rules.

- There are **FIVE** problems, totaling 50 points.
- There are extra pages after some problems. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	5	$\sum$
/7	/8	/10	/10	/15	/50

(1) (a) Define *convergence in distribution.*
(b) State the *Strong Law of Large Numbers.*

(2) Given $p \in (0,1)$ and an integer $k \geq 1$, we say that $X \sim \mathrm{PL}(k,p)$ if

$$\mathbb{P}(X = m) = \begin{cases} \binom{m-1}{k-1} p^k (1-p)^{m-k} & : \text{for } m = k, k+1, k+2, \dots \\ 0 & : \text{otherwise} \end{cases}$$

or equivalently, if

$$M_X(s) = \left[\frac{pe^s}{1-(1-p)e^s} \right]^k$$

(a) Suppose $X \sim \mathrm{PL}(k,p)$ and $Y \sim \mathrm{PL}(\ell,p)$ are independent. Determine the law of $X+Y$.

(b) Suppose $X_i \sim \mathrm{PL}(1,p)$ and $N \sim \mathrm{PL}(k,q)$ are mutually independent. Show that

$$Y = \sum_{i=1}^{N} X_i$$

is PL-distributed and determine the associated parameters.

$\boxed{\text{extra paper}}$

(3) Let X be a random variable with

$$M_X(s) = \begin{cases} (1-s^2)^{-1} & : \text{for } -1 < s < 1 \\ \infty & : \text{otherwise.} \end{cases}$$

(a) Use the Chebyshev inequality to give a bound on $\mathbb{P}(|X| \geq 10)$.
(b) Use the Cramer–Chernoff method to give a bound on $\mathbb{P}(X \geq \frac{3}{4})$.

$\boxed{\text{extra paper}}$

(4) (a) State the Central Limit Theorem.
(b) Suppose $X \sim \text{Binomial}(100, 1/2)$. Use a CLT approximation to estimate

$$\mathbb{P}(40 \leq X \leq 55).$$

Remember to include the DeMoivre–Laplace correction.

$\boxed{\text{extra paper}}$

(5) Let $X_n \sim N(0, \sigma_n^2)$ be independent random variables.

(a) Show that $X_n \to 0$ in L^2 sense if and only if $\sigma_n \to 0$.

(b) Show that if $\sum \sigma_n^2 < \infty$, then $X_n \to 0$ with probability one.

$\boxed{\text{extra paper}}$