**Definition.** A sequence of random variables $X_n$ is said to *converge almost surely* to a random variable $X$ if

$$\mathbb{P}\big(\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\big) = 1.$$

**Remark.** The adverbs *almost surely* and *with probability one* can be used interchangeably to mean that the thing stated happens on an event of probability one. In this case it is the convergence of the sequence $X_n$.

Our first application of the ideas reviewed above is the proof of the following:

**Proposition.** *Let $X$ and $X_n$, $n \in \mathbb{N}$, be random variables. If $X_n \to X$ almost surely, then $X_n \to X$ in probability.*

**Remarks.** 1. The converse is false; convergence in probability does *not* imply convergence almost surely. However, if $X_n \to X$ in probability, then there is a *subsequence* $X_{n_k}$ that converges to $X$ almost surely.

2. Convergence in $L^p$ implies convergence in probability and thus convergence of a subsequence almost surely. However, convergence almost surely does *not* guarantee convergence in $L^p$ for any $p$, not even along a subsequence.

*Proof.* Given $\epsilon > 0$, consider the events

$$A_n = \{\omega : |X_m(\omega) - X(\omega)| > \epsilon \text{ for some } m \geq n\} = \bigcup_{m=n}^{\infty} \{\omega : |X_m(\omega) - X(\omega)| > \epsilon\}$$

Evidently, the events $A_n$ are decreasing and thus

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(A_n) &= \mathbb{P}\Big(\bigcap_{n=1}^{\infty} A_n\Big) \\ &= \mathbb{P}\big(\{\omega : |X_m(\omega) - X(\omega)| > \epsilon \text{ for infinitely many } m\}\big)\end{aligned}$$

But clearly this last event can only occur at points $\omega$ where convergence fails. Thus

$$\lim_{n\to\infty} \mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}) \leq \lim_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(\{\omega : \text{ convergence fails}\}) = 0,$$

which proves convergence in probability. $\square$

**Proposition.** *Let $X$ and $X_n$, $n \in \mathbb{N}$, be random variables. If*

$$\sum_n \mathbb{E}\Big\{|X - X_n|^p\Big\} < \infty,$$

*for some $p \geq 1$, then $X_n \to X$ almost surely.*

*Proof.* Consider the events indexed by $j \in \mathbb{N}$,

$$B_j = \{\omega : |X_n(\omega) - X(\omega)| < 2^{-j} \text{ for all but finitely many } n\}.$$

Note that by the definition of convergence of a sequence,

$$\{\omega : X_n(\omega) \to X(\omega)\} = \bigcap_{j=1}^{\infty} B_j$$

Thus to prove the proposition, we need only show that all events $B_j$ have probability one. On the other hand, by the Borel–Cantelli lemma, this follows if we can prove

$$\sum_n \mathbb{P}\left(\{\omega : |X_n(\omega) - X(\omega)| \geq 2^{-j}\}\right) < \infty.$$

That is our goal.

The Markov inequality shows us that

$$\mathbb{P}\left(\{\omega : |X_n(\omega) - X(\omega)| \geq 2^{-j}\}\right) \leq \frac{\mathbb{E}\{|X_n - X|^p\}}{2^{-jp}}$$

and so,

$$\sum_n \mathbb{P}\left(\{\omega : |X_n(\omega) - X(\omega)| \geq 2^{-j}\}\right) \leq 2^{jp} \sum_n \mathbb{E}\{|X_n - X|^p\} < \infty.$$

This fulfills our stated goal and so completes the proof. □

**Exercise.** Use the preceding proposition to show the following fact that was mentioned in the remarks above: If $X_n \to X$ in $L^p$ sense, then there is a subsequence $X_{n_k}$ that converges to $X$ almost surely.