(1) Suppose X and Y are independent and both are uniformly distributed on $[0,1]$. What is the pdf of $X+Y$?

(2) Recall from class that given a vector $\vec{\mu} \in \mathbb{R}^2$ and a 2×2 positive definite matrix Σ, we say that $(X_1, X_2) \sim N(\vec{\mu}, \Sigma)$ if they have joint pdf

$$f_{X_1,X_2}(\vec{x}) = [\det(2\pi\Sigma)]^{-1/2} \exp\big\{-\tfrac{1}{2}(\vec{x}-\vec{\mu}) \cdot \Sigma^{-1}(\vec{x}-\vec{\mu})\big\}.$$

The following questions refer to this scenario.
(a) Show that X_1 and X_2 are independent if and only if the are uncorrelated.
(b) What is the variance of $\alpha X_1 + \beta X_2$, for $\alpha, \beta \in \mathbb{R}$.

(3) Suppose now that $(X_1, X_2) \sim N(\vec{0}, \mathrm{Id})$ where Id denotes the identity matrix. Compute the pdf of $X_1^2 + X_2^2$.

(4) If $X \sim N(\mu, \sigma^2)$, find $\mathbb{E}(e^{tX})$. Hint: complete the square.

(5) We say that X has a log normal distribution with parameters μ and σ, written $X \sim \ln \mathrm{N}(\mu, \sigma^2)$, if $\ln(X) \sim N(\mu, \sigma^2)$.
(a) What are the mean and variance of X?
(b) Suppose $X_1 \sim \ln \mathrm{N}(\mu_1, \sigma_1^2), \ldots, X_k \sim \ln \mathrm{N}(\mu_k, \sigma_k^2)$ are statistically independent and represent the factor by which my investment changes over the course of successive years. What is the mean and variance of my investment after k years?
(c) Suppose now that $\mu_1 = \cdots = \mu_k = 0.05$ and $\sigma_1 = \cdots = \sigma_k = 0.05$. What is the probability that my investment has doubled (or more) in value at the end of seven years?

(6) Problem 6 from Chapter 3.

(7) A farmer has two breeds of chickens, A and B, in proportion $3:5$. The annual weight gain factor for breed A is $\sim \ln \mathrm{N}(\frac{1}{2}, \frac{1}{2})$, for breed B it is $\sim \ln \mathrm{N}(\frac{1}{2}, 1)$.
(a) What is the expected average weight gain factor across the farm?
(b) A chicken more that doubles in weight. What is the probability that is of breed A?
(c) Suppose breed A chickens have brown feathers with probability 0.5 and for breed B this probability is 0.4; moreover, feather color is independent of weight gain within each breed. If a brown chicken more that doubles in weight. What is the probability that is of breed A?

(8) Problem 21 from Chapter 3.

(9) Read Problem 33 from Chapter 3.

(10) Fix $0 < a < 1$. Let $N \sim \mathrm{Poisson}(\lambda)$. Given N we generate independent $X_1, \ldots, X_N \sim \mathrm{Uniform}(0,1)$.
(a) What is the probability distribution of the number of points $X_1, \ldots, X_N$ that lie in the interval $[0, a]$ conditioned on $N = n$?
(b) What is the (unconditional) probability distribution of the number of points $X_1, \ldots, X_N$ that lie in the interval $[0, a]$?

(11) Problem 34 from Chapter 3.

(12) Suppose

$$(X,Y) \sim N\big(\vec{0}, \Sigma = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]\big)$$

(a) What is the marginal law of Y?
(b) What is the law of X conditioned on $Y = 1$?

(13) Suppose $X \sim$ Exponential($\lambda = 1$) and the law of Y conditioned on $X = x$ is Exponential(x).
(a) What is the joint pdf of X and Y?
(b) What is the marginal law of Y?
(c) Find $\mathbb{E}(\frac{1}{1+Y})$ in two ways: via part (b) and via the total expectation theorem (see box on page 173, but reverse the roles of X and Y).
(d) What is the law of X conditioned on $Y = y$?
(e) What is $\mathbb{E}(X|Y = y)$?