170A Killip | Homework 7 | Due Monday, Nov 19

(1) We have a physical process that produces independent identically distributed random variables $X_j \sim \text{Poisson}(\lambda)$; however we are not sure of the value of λ, we wish to use our observations to estimate λ. Specifically, we consider the average of our observations

$$\bar{X} = \tfrac{1}{n}\big(X_1 + \cdots + X_n\big)$$

as a way of estimating λ.
(a) Compute $\mathbb{E}(\bar{X})$. If this is λ, we call $\bar{X}$ an *unbiased* estimate of λ.
(b) Compute the variance of $\bar{X}$. (This gives an indication of how precisely $\bar{X}$ approximates λ.

(2) (Continued from previous problem.) We know that λ is also the variance of a Poisson(λ) random variable. With this in mind, we might try to estimate λ using

$$Z = \tfrac{1}{n}\big([X_1 - \bar{X}]^2 + \cdots + [X_n - \bar{X}]^2\big)$$

(a) Prove that

$$Z = \bigg(\tfrac{1}{n}\sum_{i=1}^{n} X_i^2\bigg) - \bar{X}^2.$$

(b) Find $\mathbb{E}(Z)$. Is Z biased or unbiased as an estimator of λ?
(c) Informed by computations with the normal distribution, people are often taught to use $\frac{n}{n-1}Z$ as an estimator for the variance. Is this better?

(3) Suppose X_1 and X_2 are i.i.d. Geometric(p). What is the probability that $X_1 \geq X_2$?

(4) Recall that a geometric random variable describes the number of trials needed to obtain the first success. Let us now ask about the number of trials Y_r required to achieve exactly r successes.
(a) Compute the PMF of Y_r by direct analysis from this description.
(b) Use the definition to express Y_r as a sum of i.i.d. random variables.
(c) As a check, use your answer to (b) to verify the PMF of Y_r.

(5) The life-time T (measured in minutes) of a certain radioactive nucleus follows an Exponential law with parameter $\lambda = 1$.
(a) Given that the nucleus has not disintegrated in the first minute, what is the probability that it will disintegrate before a total of two minutes have passed?
(b) Given that $T \geq t$, what is the probability that $T \in [t, t+a]$?

(6) Now suppose I have ten nuclei of the type described in the preceding problem, whose disintegrations are statistically independent.
(a) What is the probability that all ten remain intact at some specified time t?
(b) What is the probability that exactly 3 will disintegrate before some specified time t?

(7) Fix $\nu > 0$. Consider a continuous random variable X taking values in $[0, \infty)$ according to the PDF

$$f(x) = \tfrac{1}{c_\nu} x^{\frac{\nu}{2}-1} e^{-x/2} \quad \text{where} \quad c_\nu = \int_0^\infty x^{\frac{\nu}{2}-1} e^{-x/2}\, dx$$

Use integration by parts to recursively determine $\mathbb{E}(X^n)$ for each $n = 1, 2, \ldots$.
Hint: Avoid figuring out what c_ν is!

(8) (a) For X as in the previous problem, determine $\mathbb{E}(e^{tX})$ for $t < \frac{1}{2}$ via a change of variables.
(b) Why am I restricting to $t < \frac{1}{2}$?

(9) Let X be a random variable taking positive values and suppose $\ln(X) \sim N(0,1)$.
(a) What is the pdf of X.
(b) What is the probability that $X \geq 2$?
(c) What is the probability that $\frac{1}{2} \leq X \leq 2$?
(d) For which value of λ is $\mathbb{P}(X > \lambda) = \frac{1}{10}$?

(10) Let $\Theta \sim \text{Uniform}(0, 2\pi)$. What are the pdf and cdf of $X = \cos(\Theta)$.

(11) We chose a random point on the sphere by choosing the colatitude $\Phi \in [0, \pi]$ and longitude $\Theta \in [0, 2\pi]$ according to the pdf $f_{\Phi,\Theta}(\phi, \theta) = C \sin(\phi)$.
(a) What is the value of the constant C?
(b) Are Θ and Φ independent?
(c) What is the pdf for $Z = \cos(\Phi)$?
(d) What is the pdf for $X = \sin(\Phi)\cos(\Theta)$?