170A Killip | Homework 6 | Due Fri, Nov 9

(1) Each of n people (whom we label 1, 2, ..., n) are randomly and independently assigned a number from the set $\{1, 2, 3, \dots, 365\}$ according to the uniform distribution. We will call this number their birthday. You may assume $2 \leq n \leq 365$.
(a) What is the expected number of pairs who share a birthday given that at least one pair share a birthday?
(b) Check your formula against a direct computation in the case $n = 2$.

(2) I play the following game using a coin that lands heads with probability p. I start with $X_0 = \$1$ and at each stage I gamble all I have on the toss of the coin. If it lands heads I end up with twice what I started with; if it lands tails I lose everything. All coin tosses are statistically independent.
(a) With X_n denoting how much money I have after the nth toss, find

$$\mathbb{E}(X_{n+1}|X_n = k)$$

in terms of k.
(b) Find $\mathbb{E}(X_n)$ for all n.

(3) Suppose 30% of widgets manufactured are used indoors, the remainder being used outdoors.
(a) When outdoors, the life expectency of a widgit is three seasons; express this as a conditional expectation.
(b) When indoors, the life expectency of a widgit is five seasons. What is the life expectancy of a widgit at manufacture.

(4) (a) The number of seasons we get out of a widgit installed indoors follows a Poisson distribution. Use the information in the last problem to determine the parameter λ.
(b) Given that an indoor widgit is still working two seasons after installation, what is its expected lifespan (since manufacture).

(5) My daughter repeatedly attempts to build a stack of three blocks. Her probability of balancing any particular block is p (including the first block of the stack), independent of any other attempt. Failure to balance any particular block results in the whole stack collapsing.
(a) What is the probability she successfully builds the stack on any one attempt?
(b) What is the PMF for the number of attempts needed to first successfully complete the stack.
(c) What is the (conditional) PMF for the number of blocks successfully balanced in any particular attempt to build the stack, given that the attempt fails?
(d) What is the expected number of blocks balanced succesfully in a failed attempt to build the stack?
(e) Fix $\ell \in \{1, 2, 3, \dots\}$. Given that the first successful completion of the stack happend on the ℓth attempt, what is the expected value of the total number of blocks balanced successfully over the course of these ℓ attempts?
(f) What is the expected total number of blocks balanced successfully up to and including the first successful completion of the whole stack?

(6) Suppose $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are statistically independent. Show that $X + Y \sim \text{Poisson}(\mu + \lambda)$. Hint: Use the binomial theorem.