

# THE RIGID RELATION PRINCIPLE, A NEW WEAK CHOICE PRINCIPLE

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ABSTRACT. The rigid relation principle, introduced in this article, asserts that every set admits a rigid binary relation. This follows from the axiom of choice, because well-orders are rigid, but we prove that it is neither equivalent to the axiom of choice nor provable in Zermelo-Fraenkel set theory without the axiom of choice. Thus, it is a new weak choice principle. Nevertheless, the restriction of the principle to sets of reals (among other general instances) is provable without the axiom of choice.

In this article we introduce the rigid relation principle (RR), which asserts that every set admits a rigid binary relation, or equivalently, that every set is the vertex set of a rigid directed graph. To be precise, RR asserts that for every set  $A$  there is a binary relation  $R$  on  $A$  such that the structure  $\langle A, R \rangle$  is *rigid*, meaning that it has no nontrivial automorphisms, that is, no bijective function  $\pi : A \rightarrow A$  such that  $a R b \iff \pi(a) R \pi(b)$ , other than the identity function.

The RR principle follows easily from the axiom of choice, because under AC every set  $A$  has a well-order  $<$ , and it is easy to see that well-orders are rigid: if a bijection  $\pi : A \rightarrow A$  is  $<$ -preserving, then there can be no  $<$ -minimal element  $a$  for which  $\pi(a) \neq a$ , and so  $\pi$  is the identity function. Nevertheless, we shall prove that RR is not equivalent to AC, and neither is it provable in ZF, assuming ZF is consistent. Thus, it is a weak choice principle.

**Main Theorem 1.** *The rigid relation principle RR is a weak choice principle in the sense that it is a consequence of the axiom of choice, but not equivalent to it, assuming ZF is consistent, and neither is it provable in ZF alone. Nevertheless, the restriction of RR to sets of reals (among other general instances) is provable in ZF.*

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In other words, the first sentence of the theorem asserts that ZFC proves RR, but  $ZF + RR + \neg AC$  and  $ZF + \neg RR$  are each consistent, if ZF is. The theory  $ZF + RR$  therefore lies strictly between ZF and ZFC, if these theories are consistent. This research project arose out of our inquiries and answers posted on MathOverflow at [Ham]. The claims made in the main theorem are proved separately in theorems 2, 4, 5 and 6, plus the observation above that AC easily implies RR.

Before continuing, let us mention that a related classical result of Vopěnka, Pultr and Hedrlin [VPH65] establishes in ZFC, using the axiom of choice, that every set  $A$  admits a binary relation  $R$  with what we shall call a *strongly rigid* relation, namely, the structure  $\langle A, R \rangle$  not only has no nontrivial automorphisms, but also has no nontrivial endomorphisms, that is, no functions  $f : A \rightarrow A$  with  $a R b \Rightarrow f(a) R f(b)$ , other than the identity function. No infinite well-order is strongly rigid. It is an interesting elementary exercise to see, without the axiom of choice, that every countable set has a strongly rigid relation.

### 1. THE RR PRINCIPLE IS PROVABLE IN ZF FOR SETS OF REALS

Let us begin with the observation that the rigid relation principle is outright provable in ZF when it comes to sets of reals.

**Theorem 2 (ZF).** *Every set of reals admits a rigid binary relation.*

*Proof.* We shall not use the axiom of choice in this argument. Suppose that  $A$  is a set of reals. We may regard  $A$  as a subset of Cantor space  $2^\omega$ , since this space is bijective with  $\mathbb{R}$ . We break into several cases.

If  $A$  is countable, then we may impose an order structure on it by making it a linear order isomorphic to  $\omega$ , or a finite linear order if  $A$  is finite, and these orders are rigid.

Let us suppose next merely that  $A$  has a countably infinite subset. Enumerate this subset as  $Z = \{z^*, z_0, z_1, \dots\} \subseteq A$ , where  $z^*$  and the  $z_n$  are all distinct. For each finite binary sequence  $s$ , let  $U_s$  be the neighborhood in  $2^\omega$  of all sequences extending  $s$ , so that  $U_s(x) \iff s \subseteq x$ . The structure  $\langle A, U_s \rangle_{s \in 2^{<\omega}}$  is rigid, since if a point  $x$  in  $A$  is moved to another point, then it moves out of some neighborhood  $U_s$  that it was formerly in. We now reduce this structure to a binary relation. Begin by enumerating the finite binary sequences as  $s_0, s_1$ , and so on (this does not require AC). Let  $R$  be the relation on  $A$  that, first, places all the  $z_n$  below  $z^*$ , ordered like  $<$  on  $\omega$ , and makes  $R(z^*, z^*)$  true. That is, we ensure that  $R(z^*, z^*)$  holds and that  $R(z_n, z^*)$  and  $R(z_n, z_{n+1})$  hold for every  $n$ . Next, for  $y \notin Z$ , we define that  $R(x, y)$  holds if and only if  $x = z_n$  for some  $n$  and  $U_{s_n}(y)$ . That is, the first coordinate gives you some  $z_n$ , and hence some  $s_n$ , and then you use

this to determine which neighborhood predicate to apply to  $y$ , for  $y$  outside of  $Z$ . Let us argue that the structure  $\langle A, R \rangle$  is rigid. First, observe that  $z^*$  is the only real for which  $R(z^*, z^*)$ , and that the only predecessors of  $z^*$  are of the form  $z_n$ . Each  $z_n$  is thus individually definable since  $R$  orders them in order type  $\omega$ . So  $z^*$  and each  $z_n$  are definable in  $\langle A, R \rangle$  and therefore fixed by all automorphisms. Since every  $z_n$  is fixed, it follows that every automorphism must respect the neighborhood  $U_{s_n} \cap A$ , and hence fix all reals. So there are no nontrivial automorphisms, and  $\langle A, R \rangle$  is rigid, as desired.

The remaining case occurs when  $A$  is uncountable, but has no countably infinite subset. In other words,  $A$  is infinite and Dedekind finite. In this case, every permutation of  $A$  will consist of disjoint orbits of finite length, since if there were an infinite orbit, then we could build a countably infinite subset of  $A$  by iterating it. But if every permutation of  $A$  is like that, then  $A$  has no permutations that respect a linear order, such as the usual linear order  $<$  of the reals. Thus,  $\langle A, < \rangle$  is rigid.  $\square$

It follows from this theorem that the easy observation that AC implies RR because well-orders are rigid cannot be reversed on a set-by-set basis.

**Corollary 3.** *It is relatively consistent with ZF that there are sets that are not well-orderable, but which nevertheless have rigid binary relations.*

*Proof.* The usual counterexamples to AC in the symmetric forcing models produce non-wellorderable sets of reals. (For example, consider [Jec73, Lemma 5.15]). But these sets admit rigid binary relations by theorem 2.  $\square$

Let us now generalize theorem 2 to the following result, of which we shall make use in the proof of theorem 6, where we show that RR is not equivalent to AC. We define a structure to be *hereditarily rigid* if every substructure of it is rigid. For example, every well-order is hereditarily rigid, since any suborder of a well-order is still a well-order, which is rigid. Also, every linearly ordered Dedekind finite set is hereditarily rigid, since any subset of a Dedekind finite set is Dedekind finite, and linearly ordered such sets are rigid as we mentioned in the proof of theorem 2, since any bijection of a Dedekind finite set has only finite orbits, and such a bijection, if nontrivial, cannot respect a linear order. A relation  $R$  on  $A$  is *irreflexive* if  $R(a, a)$  never holds for any  $a \in A$ .

**Theorem 4** (ZF). *If a set  $B$  has a hereditarily rigid irreflexive binary relation, then every subset  $A \subseteq \mathbb{R} \times B$  has a rigid binary relation. In particular, every subset  $A \subseteq \mathbb{R} \times \gamma$  for an ordinal  $\gamma$  has a rigid binary relation.*

*Proof.* We shall not use the axiom of choice in this argument. Suppose that  $\langle B, R_1 \rangle$  is hereditarily rigid and irreflexive, and consider  $A \subseteq 2^\omega \times B$ . As in theorem 2, we split into cases. If  $A$  is countable, then we may easily impose a rigid linear order on  $A$  as before.

For the main case, let us suppose again merely that  $A$  has a countably infinite subset  $Z = \{z^*, z_0, z_1, \dots\} \subseteq A$ , where  $z^*$  and all the  $z_n$  are distinct. Let  $s_0, s_1$ , and so on enumerate the finite binary sequences. Following theorem 2, define the relation  $R$  on  $A$  so that  $R(z^*, z^*)$  holds and so that  $R(z_n, z^*)$  and  $R(z_n, z_{n+1})$  holds for every natural number  $n$ . For  $\langle x, b \rangle \in A - Z$ , we define that  $R(z_n, \langle x, b \rangle)$  holds if and only if  $s_n \subseteq x$ , and for  $\langle x, b \rangle, \langle y, c \rangle \in A - Z$ , we define that  $R(\langle x, b \rangle, \langle y, c \rangle)$  holds if and only if  $R_1(b, c)$ . We claim that  $\langle A, R \rangle$  is rigid. To see this, suppose that  $\pi : A \rightarrow A$  is an  $R$ -automorphism. Since  $R_1$  is irreflexive, it follows that  $z^*$  is the only point for which  $R(z^*, z^*)$  holds, and so as in the proof of theorem 2, we see that  $z^*$  and every  $z_n$  are definable in  $\langle A, R \rangle$  and hence fixed by  $\pi$ . From this, it follows that  $\pi(\langle x, b \rangle) = \langle y, c \rangle$  implies  $x = y$ , since this is true inside  $Z$ , and otherwise outside  $Z$ , if  $x \neq y$ , then some  $z_n$  will be  $R$ -related to  $\langle x, b \rangle$ , but  $\pi(z_n) = z_n$  is not related to  $\langle y, c \rangle$ . Thus,  $\pi$  fixes all the first coordinates, and therefore on  $A - Z$  amounts to an  $R_1$ -automorphism of the second coordinates, on every slice of  $A - Z$ . Since every such slice is a subset of  $B$ , it remains rigid under the relation  $R_1$ . And so  $\pi$  is the identity on  $A$ , as desired.

Finally, for the remaining case, we suppose that  $A$  has no countably infinite subset. In this case, define  $R(\langle x, b \rangle, \langle y, c \rangle)$  if and only if  $x < y$  in the reals, or  $x = y$  and  $R_1(b, c)$ . If  $\pi : A \rightarrow A$  is an  $R$ -automorphism, then since  $A$  is Dedekind finite, we see that  $\pi$  consists of finite orbits. We claim that  $\pi(\langle x, b \rangle) = \langle y, c \rangle$  implies  $x = y$ . To see this, consider first the case  $x < y$ , which implies that  $\langle x, b \rangle$  is  $R$ -related to  $\pi(\langle x, b \rangle)$ , and consequently by induction each point in the orbit of  $\langle x, b \rangle$  is  $R$ -related to the next; but since the orbit is finite, these points eventually return to  $\langle x, b \rangle$  and therefore at some point must have their first coordinates strictly descend, preventing  $R$  at that step of the orbit, a contradiction. The case  $y < x$  is similarly contradictory, and so  $x = y$ , as desired. Thus,  $\pi$  amounts to an automorphism on the second coordinates of each slice of  $A$ , and since  $R_1$  is hereditarily rigid, it follows that  $\pi$  is the identity.  $\square$

## 2. THE RR PRINCIPLE IS NOT PROVABLE IN ZF

We prove next that the rigid binary relation principle is not provable in ZF, assuming ZF is consistent, and therefore it may be viewed as a nontrivial choice principle.

**Theorem 5.** *If ZF is consistent, then so is  $\text{ZF} + \neg\text{RR}$ . Specifically, every model of ZF admits a symmetric extension in which RR fails.*

*Proof.* We shall make use of the permutation model technique for constructing models of ZF set theory with violations of AC. It will be enough to show that any model of ZF can be extended to a permutation model  $M \models \text{ZFA}$ , that is, Zermelo-Fraenkel set theory with atoms, in which the set of atoms  $A$  has no rigid binary relation. To see that this suffices, suppose we have such a model  $M$ . Observe first that the statement that every binary relation on  $A$  has a nontrivial automorphism can be expressed using the  $\in$  relation and quantifiers bounded by  $\mathcal{P}^5(A)$ , the fifth iteration of the powerset operation on  $A$ . Meanwhile, the Jech-Sochor theorem [Jec73, Theorem 6.1] provides a symmetric extension  $N \models \text{ZF}$  containing a set  $B$  for which  $\langle \mathcal{P}^5(A), \in \rangle^M$  is isomorphic to  $\langle \mathcal{P}^5(B), \in \rangle^N$ . It follows therefore that RR fails in  $N$ , and we will have thus achieved our desired model.

The model  $M$  we shall present will be the basic Fraenkel model described in [Jec73, section 4.3], which is determined inside a fixed ZFA model with a countably infinite set  $A$  of atoms by the group  $\mathcal{G}$  of all permutations of  $A$  and the filter  $\mathcal{F}$  generated by the subgroups of the form  $\text{fix}(E) = \{\pi \in \mathcal{G} : \pi(a) = a \text{ for all } a \in E\}$ , where  $E$  is any finite subset of  $A$ . Note that any automorphism  $\pi$  of  $A$  extends hereditarily in the natural way from  $A$  to the universe built over  $A$  by the  $\in$ -recursive definition  $\pi(x) = \{\pi(y) \mid y \in x\}$ , and so  $\pi(x)$  makes sense for arbitrary sets in our ZFA universe with atoms  $A$ . We define that such a set  $x$  belongs to  $M$  exactly when there is some finite  $E \subseteq A$  for which  $\text{fix}(E) \subseteq \text{sym}(x)$ , where  $\text{sym}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$ . Note that when  $R$  is a binary relation on  $A$ , the set  $\text{sym}(R)$  consists precisely of the automorphisms of  $\langle A, R \rangle$ .

The resulting model  $M$  satisfies ZFA for reasons explained in [Jec73]. This model  $M$  can be realized as an extension of any model of ZF by first adjoining countably many atoms and then taking the submodel obtained by using  $\mathcal{G}$  with  $\mathcal{F}$  as described. We claim now additionally that in  $M$  there is no rigid binary relation on  $A$ . To see this, suppose  $R \subseteq A \times A$  is in  $M$ . Thus, there is a finite set  $E \subseteq A$  such that whenever  $\pi \in \text{fix}(E)$ , then  $\pi \in \text{sym}(R)$ . In other words, any automorphism  $\pi$  of  $A$  fixing the finite set  $E$  pointwise will be an automorphism of

$\langle A, R \rangle$ . Since  $A$  is infinite, we may easily find an automorphism  $\pi$  of  $A$  that swaps two atoms outside  $E$ . Since  $\pi$  fixes  $E$ , it follows that  $\pi$  is an automorphism of  $\langle A, R \rangle$ ; and since  $\pi$  has finite support, it follows that  $\pi \in M$ . Thus,  $M$  has a nontrivial automorphism of  $\langle A, R \rangle$ , and consequently RR must fail in  $M$ , as desired.  $\square$

It follows from the proof of theorem 5 and by examining the proof of the Jech-Sochor theorem that our theorem 2 is optimal in the sense that while it asserts that every set of reals admits a rigid binary relation, this cannot be improved in general to sets of sets of reals. This is because models of ZFA are simulated in the Jech-Sochor theorem by ZF models by a map transferring the set of atoms to a set of sets of reals. Specifically, the set  $B$  in the proof of theorem 2 for which  $\langle \mathcal{P}^5(A), \in \rangle^M$  is isomorphic to  $\langle \mathcal{P}^5(B), \in \rangle^N$  is a member of  $\mathcal{P}^2(\mathbb{R})$ . So although theorem 2 is proved in ZF for sets of reals, the conclusion need not hold for sets of sets of reals. (This argument also shows that the Jech-Sochor theorem cannot be improved to map the atoms into the reals, a fact which has already been known by other means.)

We call attention to the curious fact that all of our provable instances of the rigid relation principle have a hereditary nature; that is, for the sets we have shown in ZF to admit rigid binary relations in theorems 2 and 4, their subsets also admit rigid binary relations. At first it might seem possible that this is due to a general phenomenon and that it is provable in ZF that whenever a set admits a rigid binary relation, every subset will as well. This is not the case. As pointed out to us by Ali Enayat, under ZF the  $\in$  relation is rigid on any transitive set. Thus in any model of ZF where RR fails, by taking a sufficiently large initial segment  $V_\gamma$  of the cumulative hierarchy we find that  $V_\gamma$  admits a rigid binary relation but also has a subset which does not.

The same does not hold when we allow atoms in our models. In the basic Fraenkel model  $M$  of ZFA used in the proof of theorem 5, where RR fails, we have that whenever a set admits a rigid binary relation, so too does every subset. This is because in  $M$ , we claim, the sets admitting rigid binary relations are precisely the wellorderable sets. This follows from a result of Blass [Bla77], showing that any non-wellorderable set in  $M$  has an infinite subset that is equinumerous with a subset of the atoms  $A$ . In this case, the argument from theorem 5 showing that  $A$  has no rigid binary relation in  $M$  then shows that any non-wellorderable set in  $M$  admits no rigid binary relation.

## 3. RR IS STRICTLY WEAKER THAN AC

So far, we have observed that RR is provable from the axiom of choice, but not without. In this section, we prove nevertheless that RR is not equivalent to AC. So it is a strictly weaker choice principle.

**Theorem 6.** *If ZF is consistent, then so is ZF + RR +  $\neg$ AC. Indeed, RR holds in the Cohen model of ZF +  $\neg$ AC.*

*Proof.* We shall apply theorem 4 to show that RR holds in the symmetric Cohen model  $M$  of ZF +  $\neg$ AC. Symmetric models are similar to permutation models, but built instead with a group  $\mathcal{G}$  of automorphisms of a notion of forcing  $\mathbb{P}$ . For the Cohen model, let  $\mathbb{P} = \text{Add}(\omega, \omega)$  be the usual forcing notion to add  $\omega$  many Cohen reals, that is, the partial order consisting of the finite partial functions from  $\omega \times \omega$  into  $\{0, 1\}$ , ordered by reverse inclusion. Every permutation  $\pi : \omega \rightarrow \omega$  induces an automorphism  $\pi : \mathbb{P} \rightarrow \mathbb{P}$ , defined by permuting the Cohen reals in the same way, so that

$$\pi(p)(\pi(m), n) = p(m, n).$$

Let  $\mathcal{G}$  be the collection of these induced automorphisms of  $\mathbb{P}$ . Note that any automorphism of  $\mathbb{P}$  induces hereditarily a corresponding automorphism of the class of  $\mathbb{P}$ -names  $\tau \mapsto \tau^\pi$ , defined recursively by  $\tau^\pi = \{ \langle \sigma^\pi, \pi(p) \rangle \mid \langle \sigma, p \rangle \in \tau \}$ , and one may prove inductively that  $p \Vdash_{\mathbb{P}} \varphi(\tau)$  if and only if  $\pi(p) \Vdash_{\mathbb{P}} \varphi(\tau^\pi)$ . Let  $M$  be the submodel of the generic extension  $V[G]$  of elements  $\tau_G$  having names  $\tau \in V^{\mathbb{P}}$  which are symmetric in the sense that there is some finite  $E \subseteq \omega$  such that whenever  $\pi$  fixes each member of  $E$ , then  $\tau = \tau^\pi$ . Jech [Jec73, Lemma 5.25] establishes the following property of this model, which we state without proof.

**Lemma 6.1.** *In the Cohen model  $M$ , there is a set of reals  $A$  such that every set can be injected into  $A^{<\omega} \times \gamma$  for some ordinal  $\gamma$ .*

It follows from this that every set in  $M$  can be injected into  $\mathbb{R}^{<\omega} \times \gamma$  and indeed into  $\mathbb{R} \times \gamma$  for some ordinal  $\gamma$ . Thus, every set in  $M$  is bijective with a subset of  $\mathbb{R} \times \gamma$ . But theorem 4 establishes that all such sets admit a rigid binary relation, and so RR holds in  $M$ , as desired.  $\square$

This completes the proof of theorem 6 and therefore also of the main theorem stated at the beginning of the paper.

Before concluding the paper, let us make a few small remarks about how RR relates to some of the other well-known weak choice principles. Since we have proved that RR holds in the Cohen model, RR fails to imply any of the choice principles that fail in that model. For example, RR is consistent with the existence of infinite Dedekind finite sets of

reals, since such sets exist in that model. In particular, it follows that RR does not imply the axiom of countable choice or the axiom of dependent choices, as these fail in the Cohen model. Meanwhile, RR does not follow from  $AC_\omega$ , or from  $AC_\kappa$  for any cardinal  $\kappa$ , because RR fails but these axioms hold in the Fraenkel model arising from countable support (or size  $\kappa$  support) in place of finite support in the proof of theorem 5. We note that in all of the models of RR provided in this article, every set can be linearly ordered, and it would be interesting to separate this principle from RR. Also, we have not yet separated RR from the prime ideal theorem.

We close the paper by mentioning a few of the many directions in which one could hope to continue this research. It would be natural to consider the rigid relation principle in higher arities, so that one gets rigid ternary relations or rigid  $k$ -ary relations on every set. Similarly, one could consider the principles asserting that every set has a rigid first-order structure in a finite language or in a countable language. Returning to binary relations, it would be natural to consider the question of whether every set has a rigid linear order, or a rigid symmetric relation. Another way to describe this last idea is that although the rigid relation principle asserts that every set is the vertex set of a rigid directed graph, it is also natural to insist that every set is the vertex set of a rigid graph. In addition, with each of these variant rigidity principles one might also consider the notion of hereditary rigidity that we introduced above, as well as strong rigidity (no endomorphisms rather than merely no automorphisms).

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