1 Review

The most fundamental objects in set theory, other than the sets, are the ordinals.

Definition 1.1. *A binary relation R on a set A is* well-founded *if every nonempty subset $B \subseteq A$ has a minimal element; an element $b \in B$ such that for all $c \in B$, cRb fails.*

The Axiom of Foundation, which we shall always assume throughout the course, says that for any set the $\in$ relation applied to that set is well-founded. Every set has an $\in$-minimal element. In particular there is no set containing itself.

Definition 1.2. *A linear order $<$ on a set W is a* well-ordering *if it is a well-founded linear order.*

Notice that for a linear order a minimal element b of a subset B is also least, $b \leq c$ for all $c \in B$; this is simply because the definition of a linear requires that for all b, c either $b < c$ or $c < b$ or $b = c$. Thus a well-ordering is a linear order where every nonempty subset has a least element. Any finite linear order is well-ordered; the natural numbers are another example of a well-ordered set. The *Axiom of Choice* says that for any set A there is a well-ordering $<$ on A.

Definition 1.3. *A set x is transitive if every element of x is a subset of x; that is if $z \in y$ and $y \in x$ then $z \in x$. Another way of saying this is that x is closed downwards under $\in$.*

Definition 1.4. *A set α is an ordinal if it is transitive and well-ordered by the $\in$ relation.*

Given an ordinal α, one can form its successor $\alpha + 1 = \alpha \cup \{\alpha\}$ which is also an ordinal. An ordinal α is called a successor ordinal if it has the form $\alpha = \delta + 1$ for some δ. Any nonzero ordinal which is not a successor is called a *limit ordinal.*

Proposition 1.5. *Every well-ordered W set is isomorphic to a unique ordinal α. That is, there is a bijection $f : W \to \alpha$ so that $a < b$ if and only if $f(a) \in f(b)$.*

We write ON for the class of all ordinals; this class itself is well-ordered by $\in$. In particular if α, β are distinct ordinals either $\alpha \in \beta$ or $\beta \in \alpha$. If X is a set of ordinals then $\bigcup X$ is an ordinal. In particular for an increasing sequence of ordinals $\langle \alpha_i : i < \lambda \rangle$ we write $\lim_{i<\lambda} \alpha_i = \bigcup_{i<\lambda} \alpha_i$. Another way of defining the limit is to say that it is the least ordinal α such that for each $i < \lambda$, $\alpha_i \leq \alpha$. A sequence $\langle \alpha_i < \lambda \rangle$ is *unbounded* in α if for each $\beta < \alpha$ there is an $i < \lambda$ with $\beta < \alpha_i$. The sequence is unbounded in α exactly when α is its limit.

One can think of the ordinals as being built up by starting with 0, taking the next ordinal 1 to be the set $\{0\}$ containing that ordinal, and $2 = \{0, 1\}$ the set containing the previous two. We continue this way making the next ordinal the collection of all the previous ordinals. When we have done this as long as we can we will have a collection of ordinals $\{0, 1, 2, \ldots\}$. We call this collection ω. Then $\omega + 1 = \omega \cup \{\omega\}$ and we continue on in that way.

By the Axiom of Choice, any set can be well-ordered. By Proposition 1.5 for any set A there is an ordinal α and a bijection $f : A \to \alpha$. We let $|A|$ be the least such α. It follows immediately from the definition that $|A| = |B|$ if and only if there is a bijection $f : A \to B$. The following theorem makes it a little easier to construct bijections.

Theorem 1.6 (Cantor-Schroder-Bernstein)**.** *If there is an injection from A into B and an injection from B into A, then $|A| = |B|$.*

Proof. Without loss of generality A and B are disjoint; otherwise replace A by $\{0\} \times A$ and B by $\{1\} \times B$. Let $f : A \to B$ be an injection, and let $g : B \to A$ be another one. Our goal is to build a bijection $h : A \to B$.

Now, given an $a \in A$ consider the set

$$S_a = \{\ldots, f^{-1}(g^{-1}(a)), g^{-1}(a), a, f(a), g(f(a)), f(g(f(a))) \ldots\},$$

and for $b \in B$ consider the set

$$S_b = \{\dots g^{-1}(f^{-1}(b)), f^{-1}(b), b, g(b), f(g(b)), \dots\}.$$

Both sequences alternate between members of A and B and move forward by applications of f and g, and backwards by application of f^{-1}, g^{-1}. Now it may be the case that going left there is no inverse image to take. If the first element on the left belongs to A, we call the sequence A-terminating; otherwise we call it B-terminating.

Notice that if c_1, c_2 are elements of $A \cup B$ and $c_1 \in S_{c_2}$ then $S_{c_1} = S_{c_2}$.

We now define our bijection $h : A \to B$ as follows. Let $a \in A$ be given. If S_a is A-terminating, or not terminating at all, set $h(a) = f(a)$. If S_a is B-terminating, set $h(a) = g^{-1}(a)$, which is well-defined since $a \in S_a$ and cannot be the leftmost point.

Let us check that this works. First of all, the map is surjective. For consider $b \in B$. If S_b is A-terminating or not terminating at all, $b = h(f^{-1}(b))$. If S_b is B terminating then $b = h(g(b))$.

As for injectivity, suppose $h(a_1) = h(a_2)$. We want to show that $a_1 = a_2$. If S_{a_1} and S_{a_2} are both A-terminating or do not terminate at all, then we have $h(a_1) = f(a_1) = f(a_2) = h(a_2)$, so that $a_1 = a_2$ by injectivity of f. If S_{a_1} and S_{a_2} are both B-terminating then $h(a_1) = g^{-1}(a_1) = g^{-1}(a_2) = h(a_2)$, so that $a_1 = a_2$ by applying g to $g^{-1}(a_1) = g^{-1}(a_2)$. The final case is that say S_{a_2} is B-terminating and S_{a_1} is not. But clearly the definition of h gives that $h(a_1) \in S_{a_1}$ while $h(a_2) \in S_{a_2}$. So $S_{h(a_2)}$ is B-terminating while $S_{h(a_1)}$ is not. Since $h(a_1) = h(a_2)$ that is impossible. □

Without too much difficulty one can now show that $|A| \leq |B|$ if and only if there is an injection from A into B.

Definition 1.7. *An ordinal α is a cardinal if $|\alpha| = \alpha$.*

In this course our main objective is to prove the independence of the continuum hypothesis, a question of cardinal arithmetic first posed by Cantor. Here are the basic definitions of cardinal arithmetic.

Definition 1.8. *Let κ, λ be cardinals. Then $\kappa + \lambda$ is the cardinality of $\{0\} \times \kappa \cup \{1\} \times \lambda$ (any disjoint sets of cardinality κ and λ would do here).*

Definition 1.9. *Let κ, λ be cardinals. Then $\kappa \cdot \lambda$ is the cardinality of $\kappa \times \lambda$.*

It turns out that for infinite cardinals addition and multiplication are trivial; $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$. Much more interesting is cardinal exponentiation.

Definition 1.10. *Let κ, λ be cardinals. We set κ^λ to be the cardinality of the set $\{f | f : \lambda \to \kappa\}$.*

For any κ, $|\mathcal{P}(\kappa)| = 2^\kappa$; the bijection witnessing this fact is the map sending a subset of κ to its characteristic function. Additionally $2^{\aleph_0}$ is the cardinality of $\mathbb{R}$; hence $2^{\aleph_0}$ is commonly referred to as the *continuum* and if there are $2^{\aleph_0}$ many of some object we generally say there are "continuum-many" of them.

Theorem 1.11 (Cantor)**.** *For any cardinal κ, $\kappa < 2^\kappa$.*

Proof. Clearly there is an injection from κ into $\mathcal{P}(\kappa)$ and so $\kappa \leq 2^\kappa$. To show that $\kappa \neq 2^\kappa$ we must show that there is no bijection from κ into $\mathcal{P}(\kappa)$. In fact, we show there is no surjection from κ into $\mathcal{P}(\kappa)$.

Let $f : \kappa \to \mathcal{P}(\kappa)$ be any function. Let $D = \{x : x \notin f(x)\}$. We claim that D is not in the range of f. Suppose for contradiction $f(d) = D$. Is $d \in D$? By definition, $d \in D$ if and only if $d \notin D$. So we cannot have $d \in D$ or $d \notin D$. Contadiction. □

By Cantor's theorem, for any cardinal κ there is a strictly larger cardinal λ. We write κ^+ for the least cardinal above κ.

We enumerate the class of cardinals using the $\aleph$ notation; $\aleph_\alpha$ is what we write for the αth cardinal. Formally we can define these by recursion:

1. $\aleph_0 = \omega$.
2. $\aleph_{\alpha+1} = \aleph_\alpha^+$.
3. $\aleph_\delta = \lim_{\alpha<\delta} \aleph_\alpha$ (for δ a limit ordinal).

The notation ω_α is also often used for $\aleph_\alpha$, the idea being that we think of ω_α as an ordertype whereas we think of $\aleph_\alpha$ has a cardinality. Another good reason for using ω_α is that it is easier to write on a blackboard.

The main question we are mostly concerned with is:

The Continuum Hypothesis(CH). $2^{\aleph_0} = \aleph_1$.

Our goal for the course is to prove that CH is independent of ZFC; there is no way to either prove or disprove it from those axioms. In fact there is amazingly little one can actually say about $\aleph_\alpha = 2^{\aleph_0}$ on the basis of ZFC alone. There is one restriction other than Cantor's theorem; let us investigate that now.

Definition 1.12. *Let α be a limit ordinal. We say that the* cofinality *of α, written cf(α), is the least $\lambda \leq \alpha$ such that there is a sequence $\langle \alpha_i : i < \lambda \rangle$ of ordinals less than λ such that* $\lim_{i<\lambda} \alpha_i = \alpha$. *An ordinal is* regular *if cf(α) $= \alpha$. Otherwise we say that it is* singular.

Proposition 1.13. *cf(α) is always a regular cardinal.*

Proof. First note that $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$. Otherwise, suppose for contradiction there is a set $\langle i_\xi : \xi < \eta \rangle$ unbounded in $\mathrm{cf}(\alpha)$ with $\eta < \mathrm{cf}(\alpha)$. But then we can easily see $\langle \alpha_{i_\xi} : \xi < \eta \rangle$ is unbounded in α. Since $\eta < \mathrm{cf}(\alpha)$ that contradicts the definition of cofinality. So $\mathrm{cf}(\alpha)$ is regular.

Now suppose for contradiction that $\mathrm{cf}(\alpha)$ were not a cardinal. Let λ be the cardinality of $\mathrm{cf}(\alpha)$, which by assumption is less than $\mathrm{cf}(\alpha)$. There is a surjection $f : \lambda \to \mathrm{cf}(\alpha)$. Define the following increasing sequence $\langle \alpha_i : i < \lambda \rangle$ by recursion on i. Suppose inductively that $\langle \alpha_i : i < j \rangle$ has been defined. Let α_j be the limit of all the $f(i)$ for $i < j$; since $j < \mathrm{cf}(\alpha)$ we have $\alpha_j < \alpha$.

Now notice that the sequence $\langle \alpha_i : i < \lambda \rangle$ is cofinal in α; for if $\beta < \mathrm{cf}(\alpha)$, there is some $i < \lambda$ so that $f(i) = \beta + 1$, and thus $\alpha_{i+2} > \beta$. Since this sequence is shorter than $\mathrm{cf}(\alpha)$ this contradicts the definition of cofinality. ☐

Theorem 1.14 (König)**.** *For any cardinal κ, $\kappa < \kappa^{cf(\kappa)}$.*

Proof. Set $\lambda = \mathrm{cf}(\kappa)$, and fix a cofinal sequence $\langle \alpha_i : i < \lambda \rangle$. Let $A = \{f | f : \lambda \to \kappa\}$. We need to show there is no surjection from κ onto A.

If there were, we could enumerate A as $\{f_\alpha : \alpha < \kappa\}$. But now we produce a function $f : \lambda \to \kappa$ not equal to any f_α. Set $f(i)$ to be the least member of $\kappa \setminus \{f_\alpha(i) : \alpha \leq \alpha_i\}$. Since $\alpha_i < \kappa$, the set is nonempty, so the function is well defined. We claim that f is as desired. For fix an $\alpha < \kappa$; we claim $f \neq f_\alpha$. Since our fixed sequence is cofinal, there is an $i < \lambda$ such that $\alpha < \alpha_i$. Then $f(i)$ is, by definition, not equal to $f_\alpha(i)$. ☐

Since $(2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega$, this tells us that the continuum does not have cofinality ω. In particular we do not have $2^{\aleph_0} = \aleph_\omega$. This restriction on the cofinality turns out to be the only concrete restriction on the value 2^ω provable in ZFC. We will prove this eventually.

The theorems of Cantor and König above are the first examples of *diagonal arguments*. We will see many more throughout the course.