1 Notation

For those unfamiliar, we have “:=” means “equal by definition,” \( \mathbb{N} := \{0, 1, \ldots \} \) or \( \{1, 2, \ldots \} \) depending on context. (i.e. \( \mathbb{N} \) is the set or collection of counting numbers.) In addition, \( \forall \) means “for all” and \( \exists \) means “there exists.” Finally \( \subset \) means “is a subset of.” So we say \( A \subset B \) for sets \( A \) and \( B \) if \( \forall x \in A \) we have \( x \in B \) holds. BY CONTRAST, \( a \in S \) means that \( a \) is a member of \( S \).

For example, \( \{0, 1, 2\} \subset \{0, 1, 2, 3\} \subset \{0, 1, 2, 3, 4\} \) holds, and as does \( \{1\} \subset \mathbb{N} \). However, we don’t say \( 1 \subset \mathbb{N} \), but instead \( 1 \in \mathbb{N} \). Very loosely speaking, this is because \( 1 \) is not a set, but rather is an element of a set.

For me, \( \mathbb{N} \) will usually start at 0, not 1.

2 Induction

We state the following variations of the principle of mathematical induction without proof, and we will use them in this class.

**Lemma 1 (Principle of Mathematical Induction).** Let \( S \subset \mathbb{N} \). Assume that \( S \) satisfies the following properties:

1. \( 0 \in S \).
2. \( \forall j \in \mathbb{N} \). If \( j \in S \) then \( j + 1 \in S \).

Then \( S = \mathbb{N} = \{0, 1, \ldots \} \).

**Lemma 2** (Principle of Mathematical Induction starting at \( k \in \mathbb{N} \) instead of 0). Fix \( k \in \mathbb{N} \). Let \( S \subset \mathbb{N} \). Assume that \( S \) satisfies the following properties:

1. \( k \in S \).
2. \( \forall j \geq k \) such that \( j \in \mathbb{N} \), if \( j \in S \) then \( j + 1 \in S \).

Then \( \{k, k + 1, \ldots \} \subset S \).

In practice, we will want to check that some property \( P(n) \) that depends on \( n \in \mathbb{N} \) holds for a certain collection of natural numbers \( n \), usually all of them, i.e. \( \mathbb{N} \) itself. We will then check the base case, i.e. that for the starting point \( k \) (usually 0, but not always), we have that \( P(k) \) holds. We will then need to check the induction step, i.e. that if we assume \( P(j) \) holds, then we can deduce \( P(j + 1) \) holds, when \( j \geq k \). It is intuitive that this should be enough because we will have checked \( P(k) \) because we will have checked the base case. Then we know that \( P(k) \) is true, and \( P(k) \) implies \( P(k + 1) \) by using the induction step with \( j = k \). Thus \( P(k + 1) \) is true. Then using the induction step for \( j = k + 1 \) we have \( P(k + 1) \) implies \( P(k + 2) \) so that we now also know \( P(k + 2) \) is true. We keep doing this, until we get \( P(j) \) is true for all \( j \geq k \). This can be formally deduced from the above statements of the principle of mathematical induction because you just set \( S \) to be the set of \( j \) such that \( P(j) \) is true. That is, \( S := \{j \in \mathbb{N} \mid P(j) \text{ holds}\} \).

3 Examples

Let us demonstrate the above concepts with a number of examples.
Theorem 3. \( \forall n \geq 0 \sum_{i=0}^{n} 1 = n + 1 \).

Proof. Let \( S := \{ n \mid P(n) \text{ is true} \} \) where \( \forall n \in \mathbb{N}, P(n) \) is the statement that for that \( n \), we have \( \sum_{i=0}^{n} 1 = n + 1 \). Notice that \( i \) is not a variable of that statement because it is being summed over. So we do not say that \( P(n) \) holds \( \forall i \) or anything like that.

Base case: We check first that \( P(0) \) holds. So that is that \( \sum_{i=0}^{0} 1 = 0 + 1 \). The left hand side evaluates to 1 because it is a sum of 1 copy of 1, corresponding to the term for the index \( i = 0 \).

Induction step: Assume that \( j \geq 0 \), and that \( P(j) \) is true. That is that \( \sum_{i=0}^{j} 1 = j + 1 \). Then by adding 1 to both side of this equality, we obtain that

\[
1 + \sum_{i=0}^{j} 1 = 1 + j + 1 = j + 2.
\]

But the left side is just the same as \( \sum_{i=0}^{j+1} 1 \).

Thus the base case and induction step are complete, so by lemma 1 we are done.

\[\square\]

Theorem 4. \( \forall n \geq 0 \sum_{i=0}^{n} i = n(n + 1)/2 \).

Proof. Let \( S := \{ n \mid P(n) \text{ is true} \} \) where \( \forall n \in \mathbb{N}, P(n) \) is the statement that for that \( n \), we have \( \sum_{i=0}^{n} i = n(n + 1)/2 \). Notice that \( i \) is not a variable of that statement because it is being summed over. So we do not say that \( P(n) \) holds \( \forall i \) or anything like that.

Base case: We check first that \( P(0) \) holds. So that is that \( \sum_{i=0}^{0} i = 0(0 + 1)/2 \). The left hand side evaluates to 0 because it is a sum of 1 copy of 0, corresponding to the term for the index \( i = 0 \).

Induction step: Assume that \( j \geq 0 \), and that \( P(j) \) is true. That is that \( \sum_{i=0}^{j} i = j(j + 1)/2 \). Then by adding \( j + 1 \) to both side of this equality, we obtain that

\[
j + 1 + \sum_{i=0}^{j} i = j + 1 + j(j + 1)/2 = (j + 1)(j + 2)/2.
\]

But the left side is just the same as \( \sum_{i=0}^{j+1} i \). In the above display, the second equality is just algebra.

Thus the base case and induction step are complete, so by lemma 1 we are done.

\[\square\]

Theorem 5. \( \forall n \geq 4 \text{ we have that } 2^n \geq 3n \).

Proof. Let \( S := \{ n \mid P(n) \text{ is true} \} \) where \( \forall n \in \mathbb{N}, P(n) \) is the statement that for that \( n \), we have \( 2^n \geq 3n \).

Base case: \( P(4) \) is true because the statement reduces to \( 16 \geq 12 \) which is true.
Induction step: If \( j \geq 4 \) and \( P(j) \) are assumed, then we know that \( 2^j \geq 3j \) so that multiplying by 2 we obtain \( 2^{j+1} \geq 6j = 3(j + j) \geq 3(j + 4) \geq 3(j + 1) \) so that \( P(j + 1) \) holds.

Thus the base case and induction step are complete, so by lemma 2 we are done.

From now on, I will just prove the base case and the inductive step, and it’s okay if you do the same. We will understand that we are using one of the principles of mathematical induction, and that we set \( S \) to be the collection of natural numbers for which whatever claim is of interest is true.

4 \( \mathbb{Q} \)

Reading: section 1.2 in course text.

Today we will exhibit the use of the rational zeros theorem (2.2 in the text) to prove various claims about rationality.

Let us demonstrate this using Exercise 2.1 from the text.

We are to show that \( \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31} \) are all not rational numbers.

In order to address every case other than \( \sqrt{24} \), I will show that for all \( p \) prime (3, 5, 7, 31 are all prime) we have that \( \sqrt{p} \) is not rational. We prove this claim in the following theorem. We do so by proof by contradiction, which is the method in which you assume the desired conclusion is false, and show that it is impossible because this assumption leads to a contradiction. Recall that what it means for a natural number \( p \) to be a prime is that only 1 and \( p \) are divisors of \( p \). We take for granted that every rational number can be reduced to simplest form, where its denominator and numerator share no common factors.

**Theorem 6.** Let \( p \in \mathbb{N} \) be a prime number. Then \( \sqrt{p} \notin \mathbb{Q} \).

**Proof.** Let \( p \in \mathbb{N} \) be a prime number. Suppose, for a contradiction, that \( \sqrt{p} \) is rational. Then consider the polynomial \( x^2 - p \). We have that \( p \geq 0 \) because \( p \) is a natural number. We therefore have that \( \sqrt{p}^2 = p \). Thus, subtracting \( p \) from both sides, we see that \( \sqrt{p}^2 - p = 0 \). This shows that \( \sqrt{p} \) is a solution to the polynomial equation \( x^2 - p = 0 \). Since we assumed that \( \sqrt{p} \) is a rational number, we may as well assume \( \sqrt{p} = c/d \) with \( c, d \) integers without a common divisor and \( d \neq 0 \). Then from theorem 2.2, we learn that \( c \) divides \( p \) and \( d \) divides 1. So \( c \) must be either \( -p, -1, 1, \) or \( p \). The reason the negative possibilities are included is because \( c \) is only known to be an integer, not necessarily a natural number. Similarly, \( d \) must be 1 or \( -1 \). So the only possibilities for \( c/d \) are \( -p, -1, 1, \) or \( p \). But \( c/d = \sqrt{p} \) so that we have either \( -p = \sqrt{p} \) or \( -1 = \sqrt{p} \) or \( 1 = \sqrt{p} \) or \( p = \sqrt{p} \). But all of these are false, so that it cannot be that any one of them holds true. Thus, we have our contradiction in that we have proven both that none of those 4 equations holds true, and also that at least one of them does. Thus, our original assumption that \( \sqrt{p} \) is rational must be false, so it must be irrational. \( \square \)
In any proof by contradiction, it is good practice to say “Suppose, for a contradiction,” and then write whatever the opposite (negation) of the statement you want to prove is. Proof by contradiction is used to prove If-then statements. If I want to prove that $A$ implies $B$ using proof by contradiction, then I assume $A$ and not $B$ and arrive at a contradiction. That is, I assume $A$ and not $B$, and find some statement $C$ for which I can prove both $C$ and not $C$. In the example above, this statement $C$ was that “either $-p = \sqrt{p}$ or $-1 = \sqrt{p}$ or $1 = \sqrt{p}$ or $p = \sqrt{p}$.”

What remains is to address the case of $\sqrt{24}$. We will again use proof by contradiction.

**Theorem 7.** $\sqrt{24} \notin \mathbb{Q}$.

*Proof.* Suppose for a contradiction that $\sqrt{24} \in \mathbb{Q}$, so we can find $c, d \in \mathbb{Z}$ with $d \neq 0$ such that $c$ and $d$ share no common factors and $c/d = \sqrt{24}$. Then again we have that $\sqrt{24}$ solves the polynomial equation $x^2 - 24 = 0$. Therefore, by theorem 2.2 we must have that $c$ divides 24 and $d$ divides 1. So $d$ is again only either 1 or $-1$ while $c$ can be any of $-24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, 24$.

So the only possibilities for the value of $c/d$ are

$-24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, 24$.

(If $d = 1$, I copy the list for $c$, and if $d = -1$ I copy the list for $c$ in reverse but that’s the same result.) It is easy, but tedious, to check that none of these possibilities for $c/d$ actually solve the equation $x^2 - 24 = 0$. For example, $(-6)^2 - 24 = 12 \neq 0$. Please check this yourself. Thus, our contradiction is that $c/d = \sqrt{p}$ solves the equation $x^2 - 24 = 0$, but it also does not solve that equation. \hfill \square

## 5 Ordered Fields

Reading: Section 1.3 in text.

We will demonstrate a use of induction, together with the concepts of this section in the following solution to problem 3.6 from the textbook.

We can address (a) and (b) together since (a) is a special case of (b) with $n = 3$, so we only really need to prove (b).

**Theorem 8.** $\forall n \in \mathbb{N}$ and $\forall a_1, \ldots, a_n \in \mathbb{R}$ we have that $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.

*Proof.* We proceed by induction.

For the base case, $n = 1$ we have that indeed $\forall a_1 \in \mathbb{R} \ |a_1| = |a_1|$ so that we definitely have $|a_1| \leq |a_1|$.

For the induction step, assume that the claim is true for $n$ and try to prove it for $n + 1$. To do that, we let $a_1, \ldots, a_{n+1}$ be given. Consider the list of numbers $b_1 = a_1, \ldots, b_{n-1} = a_{n-1}$ but we have $b_n = a_n + a_{n+1}$. Thus we have that $a_1 + \cdots + a_{n+1} = b_1 + \cdots + b_n$. Observe that $b_1, \ldots, b_n \in \mathbb{R}$. Thus we may
use the assumed induction hypothesis (i.e. the claim for case \( n \)) in order to conclude that \(| b_1 + \cdots + b_n | \leq | b_1 | + \cdots + | b_n |\). Thus we have \(| a_1 + \cdots + a_{n+1} | = | b_1 + \cdots + b_n | \leq | b_1 | + \cdots + | b_n | = | a_1 | + \cdots + | a_{n-1} | + | a_n + a_{n+1} |\). Using the usual triangle inequality on the last member and using transitivity of \( \leq \) we obtain that \(| a_1 + \cdots + a_{n+1} | \leq | a_1 | + \cdots + | a_{n-1} | + | a_n | + | a_{n+1} | = | a_1 | + \cdots + | a_{n+1} |\). Thus we have proven the claim in case \( n + 1 \).

So by induction, we have proven the claim for all values of \( n \).

Now I’d like to take the opportunity to discuss proof by contrapositive before we move to our next example. Suppose I want to prove \( P \) implies \( Q \). To do this, we have covered two methods. Direct proof: we assume \( P \) and then try to see if we can use logical and mathematical steps to reach \( Q \). Proof by contradiction: We assume \( P \) and assume that \( Q \) is false. We then try to try to reach a contradiction. I.e. we try to find some statement \( R \) that we can show to be both true and not true.

A third method we can use to prove this type of implication statement, called proof by contrapositive, is to assume that \( Q \) is false, and then prove that \( P \) is false. Actually, this is a special case of proof by contradiction. After all, we may view this method the following way. Suppose that we set out to prove \( P \) implies \( Q \) by contradiction. Remember this means I assume \( P \) and not \( Q \). If I’ve already shown that not \( Q \) implies not \( P \), then I can logically deduce that not \( P \) must be true, i.e. that \( P \) is false. So my contradiction is that I have assumed \( P \), but also proved that \( P \) is false. Thus, by proof by contradiction, I have verified that \( Q \) follows from \( P \). It is in this way that the validity of proof by contradiction can be seen to verify the method of proof by contrapositive.

Another way you can see the validity of proof by contrapositive, or really the validity of any proof technique, is to use truth tables. (We will discuss in class the truth tables of the unary operator “not” as well as the binary operators “AND, OR, IMPLIES, and EQUIVALENT.”)

Now, I will showcase this new method of proof by contrapositive in solving problem 3.8 of the textbook. Also featured in this proof will be the discussion from last week about how to properly negate statements, since we need to negate both \( P \) and \( Q \) to use proof by contrapositive.

**Theorem 9.** Let \( a, b \in \mathbb{R} \). Show if \( a \leq b_1 \) for every \( b_1 > b \) then \( a \leq b \).

**Proof.** Fix \( a, b \in \mathbb{R} \). Our statement \( P \) is that \( \forall b_1 > b \) we have \( a \leq b_1 \) and our statement \( Q \) is that \( a \leq b \). In order to use proof by contrapositive, we should therefore assume that it is not the case that \( a \leq b \), which leads us to \( a > b \) as per the property of ordered fields. (O1, p. 14) We must strive to prove not \( P \), which is “\( \exists b_1 > b \) such that \( a > b_1 \). (Here we have properly negated the statement, and also again used O1.)

So now we know what we have to do. Let’s do it. So we are assuming \( a > b \) and we need to find a \( b_1 > b \) with \( a > b_1 \). I claim that \( b_1 = (a + b)/2 \) will do the trick. \( (a + b)/2 > (b + b)/2 = b \), and \( (a + b)/2 < (a + a)/2 = a \) by using again the properties of ordered fields. This completes the proof by contrapositive.
6 The Completeness Axiom (Sups and Infs)

Let us discuss a solution to problem 4.5 in the book. It asks us to show that if $S$ is a nonempty subset of $\mathbb{R}$ that is bounded above, and $\sup S \in S$ then $\sup S = \max S$. Let me caution you that implicitly, you are therefore expected to prove that $S$ has a maximum element. This means you must find an element $m \in S$ such that $\forall s \in S m \geq s$. Then after that, you would have to prove that $m = \sup S$. It is always like this: if an equality that we ask you to prove involves expressions like limits or maximums that don’t always exist, then you have to show that they exist in the context of the problem, and THEN prove the equality. Same with inequalities.

So our modified claim is the following:

**Theorem 10.** If $S$ is a nonempty bounded subset of the real line, then $\sup S$ exists. Furthermore, if $\sup S \in S$, then $\max S$ exists, and we have that $\sup S = \max S$.

Notice how, in order to be completely rigorous, we have to interject, within the claim, the existence of $\sup S$ and the existence of $\max S$.

**Proof.** Let $S$ be a nonempty bounded subset of the real line. Then $S$ is necessarily bounded above, because bounded implies both bounded above and bounded below. Then by the completeness axiom, $S$ has $\sup S \in \mathbb{R}$. If we further assume that $\sup S \in S$, then let $m := \sup S$. I claim that $m$ is a maximum of $S$, hence the unique maximum. It is a maximum because if $s \in S$ then by the property that $\sup S$ is an upper bound for $S$, we must have $s \leq \sup S = m$. Also, since $m = \sup S \in S$ by definition of $m$ and assumption, we must have that $m \in S$. Hence $m$ is a maximum of $S$. Maximums are always unique because if $m'$ is another maximum, then we have $m \leq m'$ because $m'$ is a maximum and $m \in S$. But by reversing the roles we’d also have $m' \leq m$ so that $m' = m$. So maxima are unique, and $m = \sup S$ is therefore the unique maximum. This verifies both the existence of the maximum, and THEN also the required equality that $\sup S = \max S$.

7 A First Introduction to Limits

The relevant section is section 7 of the course text.

Since the definition of limit is a little complicated, I will remind you of what it is.

**Definition 1.** We will say that a sequence $s_n$ of real numbers converges to the real number $s$ provided that $\forall \epsilon > 0 \exists N$ such that $\forall n > N$ we have $|s_n - s| < \epsilon$.

In the book, we show that limits, when they exist, are unique. That is to say that if $s$ and $t$ are both limits of the same sequence $s_n$ then we must have $s = t$. Thus, it makes sense to make the following definition.
Definition 2. Fix a sequence $s_n$ of real numbers. The unique limit of the sequence $s_n$, when it exists, is denoted by $\lim_{n \to \infty} s_n$.

We will first approach limits in the way you have likely done in the past, which is without rigorous proof. Let us first do exercise 7.5, and then exercise 7.4 as an example of this.

7.5(a):
\[
\sqrt{n^2 + 1} - n = [(n^2 + 1) - n^2]/(\sqrt{n^2 + 1} + n) \text{ by rationalizing the denominator.}
\]
The numerator of this last expression is just 1 and so we end up needing to evaluate the limit of $1/(\sqrt{n^2 + 1} + n)$ which is 0 because the denominator clearly gets arbitrarily large as $n$ increases.

(b):
\[
\sqrt{n^2 + n - n} = [(n^2 + n - n^2)/(\sqrt{n^2 + n} + n)] = n/(\sqrt{n^2 + n} + n) = 1/(\sqrt{1 + 1/n} + 1) \text{ using the same rationalization trick. From the last quantity, it is clear that the limit is 2 because the 1/n just disappears in the limit.}
\]

(c):
Again we use the same rationalization trick. This time, it shows that
\[
\sqrt{4n^2 + n - 2n} = (4n^2 + n - 2n)/((\sqrt{4n^2 + n} + 2n) = n/(\sqrt{4n^2 + n} + 2n)) = 1/(\sqrt{4 + 2/n} + 2) \text{ from which it follows similarly to above that the limit is 1/(\sqrt{4} + 2) = 1/4.}
\]

Notice that parts (b) and (c) were a little different in (a) because of the steps where I had to divide both numerator and denominator by $n$. That was useful because if both the numerator and denominator go to infinity, then we need to see which one goes to infinity faster. In (a), the numerator was already 1 so no adjustment was necessary.

7.4 (a):
I claim that for each $n \in \mathbb{N}$ (here it means natural numbers starting at 1) we have $\sqrt{2}/n \notin \mathbb{Q}$. This follows because, if for a contradiction, we had that there exists some natural number $n$ for which $\sqrt{2}/n \in \mathbb{Q}$ then we would have that there exists some $n \in \mathbb{N}$ and some $q \in \mathbb{Q}$ such that $\sqrt{2}/n = q$ so that $\sqrt{2} = nq$ so that $\sqrt{2}$ is rational. But we know that is not true, hence the contradiction. So the sequence $s_n = \sqrt{2}/n$ is a sequence of irrational numbers. Its limit is 0 because the numerator is fixed while the denominator goes to 0.

(b):
Look at the decimal approximations for $\sqrt{2}$. So we are talking about $s_1 = 1$, $s_2 = 1.4$, $s_3 = 1.41$, etc. As you increase how many places you go out, the approximation gets closer and closer, converging to its true limit $\sqrt{2} \notin \mathbb{Q}$. On the other hand, each of the individual $s_n$ for fixed $n \in \mathbb{N}$ are rational because they are terminating decimals. For example, $1.41 = 141/100$ and one can see the rest are rational using a similar fraction with denominator a power of 10.

8 Cauchy Sequences and Monotone Sequences

We will follow the book’s presentation in Section 10.

Let us do examples.

7
Exercise 10.6:
(a) Let $s_n$ be a sequence such that $\forall n \in \mathbb{N}$ we have $|s_{n+1} - s_n| < 2^{-n}$. Prove that $s_n$ is Cauchy, hence a convergent sequence.

Solution: Let $\epsilon > 0$ be given. Find $N \in \mathbb{N}$ so that $2^{-N} < \epsilon$. This is possible because we may take $N$ to be, for instance, larger than $\log(1/\epsilon)$. Then, if $n, m > N$, we have that $|s_n - s_m| = |\sum_{j=\max\{m,n\}+1}^{\min\{m,n\}-1} s_j - s_j| \leq \sum_{j=\min\{m,n\}}^{\max\{m,n\}-1} |s_{j+1} - s_j| \leq \sum_{j=\min\{m,n\}}^{\max\{m,n\}-1} 2^{-j} \leq 2^{-\min\{m,n\} + 1} \leq 2^{-N} < \epsilon$. Here we have used the triangle inequality and the formula for a sum of powers.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < 1/n$ for all $n \in \mathbb{N}$?

Solution: No. Consider the sequence $s_n = \sum_{j=1}^{n} \frac{1}{2^j}$. We have that $\forall n \in \mathbb{N}$, $|s_{n+1} - s_n| = \left| \sum_{j=1}^{n+1} \frac{1}{2^j} - \sum_{j=1}^{n} \frac{1}{2^j} \right| = \left| \frac{1}{2^{n+1}} \right| < 1/n$. On the other hand, by the integral test, which we will learn later, $s_n \to \infty$.

This is the end of the solution to exercise 10.6. Let me remark, for purely intuition, that the important difference about (a) and (b) is that the rate of decay in (a) has finite sum as a series, but not in (b). Notice that I did not make this remark as part of the proof, because it is irrelevant. Making intuitive remarks during a proof can suggest, in introductory courses, that you do not know which parts of what you wrote are required to make the proof rigorous.

Exercise 10.8:
Let $s_n$ be an increasing sequence of positive numbers and define $\sigma_n = 1/n \sum_{j=1}^{n} s_j$. Prove that $\sigma_n$ is also increasing.

Solution: Let $n \in \mathbb{N}$. We calculate that $\sigma_n = 1/n \sum_{j=1}^{n} s_j = 1/(n+1) \sum_{j=1}^{n+1} s_j/n = 1/(n+1) \sum_{j=1}^{n+1} (1/n) s_j = 1/(n+1) \sum_{j=1}^{n+1} (s_j + s_j/n) \leq 1/(n+1) \sum_{j=1}^{n+1} (s_j + s_{n+1}/n) = 1/(n+1) \sum_{j=1}^{n+1} s_j + \sum_{j=1}^{n+1} s_{n+1}/n = 1/(n+1) \sum_{j=1}^{n+1} s_j + s_{n+1} = 1/(n+1) \sum_{j=1}^{n+1} s_j = \sigma_{n+1}$.

Exercise 10.9:
Let $s_n$ be a sequence recursively defined by the initial condition $s_1 = 1$ and the recursive equation $\forall n \in \mathbb{N}, s_{n+1} = \frac{n}{n+1} s_n^2$.

(a) Find $s_2, s_3, s_4$.

Solution: Since $s_1$ is 1, we plug that into the recursion for $n = 1$ to obtain $s_2 = 1/2$. We then plug this into the recursion equation for $n = 2$ to get $s_3 = 1/6$. Then we set $n = 3$ and use the value of $s_3$ to obtain that $s_4 = 1/48$.

(b) Show that $\lim_{n} s_n$ exists.

Solution: We will show that $s_n$ is bounded and monotone decreasing. First we claim the following

Lemma 11. $\forall n \in \mathbb{N}$ we have $s_n \in [0, 1]$.

Proof. We proceed by induction. The base case is that $s_1 \in [0, 1]$ which is true. For the induction step, if $s_n$ is in $[0, 1]$ then so is $s_n^2$ and therefore because $\frac{n}{n+1}$ is as well, then we get $s_{n+1}$ is in $[0, 1]$.

Done by Induction.

For numbers $a \in [0, 1]$, we have $a \geq a^2 \geq \frac{n}{n+1} a^2$ whenever $n \in \mathbb{N}$. Thus, $s_n$ is also decreasing. The lemma shows that $s_n$ is bounded, by 1 say. We are done by theorem 10.2.
(c) Prove that \( \lim_{n} s_n = 0. \)
Solution: Give class time to work this out.