1 Page 208 (4.10)

Page 208 (4.10) requires justification. I already see that both sides of that equation are contractive strongly continuous semigroups on $L^2(\nu)$ so all that remains is to check that they have the same generator. Let $\Omega'_\nu$ be the generator associated to the RHS of (4.10). $\Omega_\nu$ is already the generator associated to the LHS of (4.10). Remember that $G$ is the spectral measure for $-\Omega_\nu$.

Suppose that $f \in D(\Omega_\nu)$. Then that means that $f \in D_{jd}$ where this is computed relative to the resolution $G$ of the identity on $[0, \infty)$. (Notice that here $f$ is no longer playing the role of the measurable map in Rudin, but rather the element in the Hilbert space, which is a real class in $L^2(\nu)$ here.) Then if $g \in L^2(\nu)$ then by dominated convergence applied to signed measures in the obvious way we have that $\langle \Psi(\exp(-t.))f, g \rangle - \langle f, g \rangle / t \to -\Omega_\nu(f), g \rangle$ where the $\Psi$ is computed relative to $G$ and the limit is as $t \downarrow 0$. Thus we have $\langle (\Psi(\exp(-t.)) - I)/t, f, g \rangle \to \langle \Omega_\nu(f), g \rangle$ as $t \downarrow 0$. But we also have $\forall t > 0, ||(\Psi(\exp(-t.)) - I)/t||^2 = \int_{[0, \infty)}(\exp(-t\lambda) - 1)/tdG_{f, f}(\lambda)$ which converges as $t \downarrow 0$, by DCT and the fact that $f \in D_{jd}$, to $\int_{[0, \infty)} -\lambda dG_{f, f}(\lambda) = || - \Psi(id)(f)||^2 = ||\Omega_\nu(f)||^2$. Thus we actually have that $(\Psi(\exp(-t.)) - I)/t f \to \Omega_\nu(f)$ in $L^2(\nu)$, which shows that $f \in D(\Omega'_\nu)$ and $\Omega'_\nu(f) = \Omega_\nu(f)$. Thus, we have shown that $\Omega_\nu \subset \Omega'_\nu$. But both are generators, hence they must be equal. (This last part isn’t about the self-adjointness. In general in Ethier context (or Liggett for that matter) generators cannot strictly extend each other.)

Remember the meaning of integrals against $dG$ as it appears in (4.10). Here technically no further clarification is needed because I only like to write down such an integral when it means that the LHS and RHS are equal upon being sandwiched between two Hilbert space elements for $\mathbb{R}$ or $\mathbb{C}$ case, and here that is indeed what it means.

2 p. 202 theorem 3.14 clarifications

Throughout I will still use 1 and 0 to denote his $+/−$. In this setting let us say that we are dealing with the lattice $\mathbb{Z}^2 + (1/2,1/2)$ with the usual graph structure, and that a line (segment) is an edge in this graph and that a contour is a cycle in this graph. In this context, cycle will mean the ordered list of vertices and edges between them in the usual sense, whereas image will refer to just the collection of vertices unordered through which the cycle passes.

Associated to each contour is the unit speed (not smoothly parameterized) Jordan curve in $\mathbb{R}^2$. The Jordan interior of that curve is what we will call the interior of the contour. Notice that this does not depend on the orientation or starting point of the contour in the obvious way. Given the image of a contour, therefore, I can decide what the interior is in the obvious way. We will say that two contours are disjoint if the edges within one are all distinct from the edges within the other.

We will say that a point in $\mathbb{Z}^2$ is surrounded by a contour iff it’s in the interior of the corresponding Jordan curve. This only depends on the image of
the contour again, and as before we know this is equivalent to the Jordan curve having winding number $+/1$ in either the complex analysis integral sense or equivalently the Burckel sense.

Some points to be careful about: the image of a contour does not include the edges, and hence not the graph information. We will not refer to the unordered list of vertices and edges in a contour, but we will refer to it as a subgraph of $\mathbb{Z}^2 + (1/2, 1/2)$ by using the phrase “trace of a contour.” Also, the interior of a contour is currently an open subset of $\mathbb{R}^2$. I will refer to the “interior lattice” when I need to consider the intersection of the interior with $\mathbb{Z}^2$, and there will be no need to consider the intersection of the interior with $\mathbb{Z}^2 + (1/2, 1/2)$. Interior lattices are always nonempty.

The fact is that I’m about to consider the question of representing a subgraph of $\mathbb{Z}^2 + (1/2, 1/2)$ as the union of traces of disjoint contours. (As opposed to the disjoint union of contour traces which would not be true because the relevant vertex sets can intersect) Thus, it would be natural to ask the rather unrelated question at this point which kinds of sets in the plane are images of curves, contours in the usual sense (not as in here), rectifiable curves, etc. But I won’t resolve this question.

The length of a contour is defined as the number of edges in it. This can be discerned from only the image of a contour in a well-defined way, and coincides with the length of the corresponding Jordan curve which happens to be rectifiable. The definition above of disjoint contours may not be most natural for this context because it seems natural to require the interiors to be disjoint. But doing it this way is necessary for the claim we promised of representing a certain subgraph of $\mathbb{Z}^2 + (1/2, 1/2)$ as a union of the traces of disjoint contours.

**Lemma 1.** The only contours that have precisely one interior lattice point are the contours that have image of corresponding Jordan curve a unit square.

**Proof.** Obvious.

**Lemma 2.** The interior lattice with the induced graph structure is connected.

**Proof.** Follow an arc in the topological sense, after perturbing it to avoid all lattice points.

**Theorem 3.** given two points in $\mathbb{Z}^2$ and a contour, if there exists a path between the two points whose image of the corresponding unit parameterized rectifiable path intersects at exactly one point the corresponding Jordan curve of the contour, then precisely one of the two points is in the interior, and hence the interior lattice.

**Proof.** We may assume that the path from above is of length 1. This is due to Kyle Kinneburg, and can be found in my email without me fixing it up so that it displays well in LaTeX. Okay, I think I have an argument for this Jordan curve thing. Suppose, for a contradiction, that we start in the interior, pass through the boundary, and end in the interior. Let $y$ denote the point we pass through on the boundary. We can then find a sequence of points "before" $y$, call them
Similarly, we can find a sequence of points "after" $y$, call them $z_n$, that also lie in the interior and approach $y$ (indeed, if any point "after" $y$ was in the exterior, we could connect it to infinity in the exterior, and therefore get a path from the endpoint of our original path to infinity).

Now, the fact that there is a homeomorphism from the closed disk to the closure of our domain (this is Caratheodory’s theorem), implies the following fact: there are arcs, $\gamma_n$, in the interior which join $x_n$ to $z_n$, and whose diameters go to zero, as $n$ gets large (because $x_n$ and $z_n$ are getting close). In particular, $\gamma_n$ does not intersect the boundary. But consider what this means for large $n$: take the two squares bordering the segment on the boundary containing $y$. The left square contains $x_n$ and the right square contains $z_n$, both of which are very close to $y$, but they can be joined by a curve which avoids the segment. Contradiction to the fact that this segment separates the two squares.

To see that two exteriors are also impossible, we cannot use stereographic projection as that ruins the combinatorial structure, although perhaps the above proof works in some more general setting. Anyhow, we can instead do Sam’s trick. If the two exterior points are separated by a vertical segment, then find the highest horizontal edge, which must exist and be higher than the two exterior points. Then delete that segment, replacing it with a move up, going all the way around the entire curve with a big box, then going back down 1 unit to reconnect. This results in a new Jordan curve that has both the old exterior points as interior points, as one can see using complex analysis.

Here are some failed attempts

**Lemma 4.** For every contour, there exists an interior lattice point that can be deleted so that the resulting set is the interior lattice of some contour. In fact, there exists more than one such an interior lattice point provided that the contour is not a unit square in the above sense.

**Proof.** Pick any interior lattice point, possible because there is at least one. Find an interior lattice point of maximum graph distance away from this base point. This can be deleted.

Given a finite subset $A$ of $\mathbb{Z}^2$ there exists a finite set of disjoint contours whose union of the (therefore disjoint) interior lattices is $A$. To establish this existence claim it is useful to consider complex integration and induction. May also want to consider proving the claim that interior lattices are connected in the graph theoretic sense. (To see this, see that the interior is path connected, and then record which squares it goes through in the $(1/2,1/2)$ offset lattice, these squares having centers of the interior lattice. May need to perturb the path so as to not go through corners of these offset squares.) May also be useful to prove that given two points in $\mathbb{Z}^2$ and a contour, if there exists a path between the two points whose image of the corresponding unit parameterized rectifiable path intersects at exactly one point the corresponding Jordan curve of the contour, then precisely one of the two points is in the interior, and hence...
the interior lattice. The finiteness is important because “contours can’t wrap around infinite sets.” To prove this, it pays first to see that the only contours that have one interior lattice point are the unit squares, which is clear. Then to prove this induct on the number of interior lattice points. We may assume that the path from above is of length 1, and that it passes horizontally through the contour which is vertically oriented. (All these made precise in the obvious way.) Find an extreme square in that direction and modify the contour to exclude it. This does not affect the membership of the original interior lattice points, but it allows us to use the induction hypothesis. The number of disjoint contours used is unique, and the choice of disjoint contours are unique up to permutation among the contours, reorienting of individual contours, and change of starting point of individual contours. Equivalently, the (unordered) collection of images or traces of the contours is uniquely determined. The boundary of the Jordan interior of a Jordan curve in \( \mathbb{R}^2 \) is always the image of that Jordan curve. Therefore, the boundary of the interior of a contour is always equal to the image of the Jordan curve corresponding to that contour. Therefore, here the union of the images of the corresponding Jordan curves is well-defined, and equal to the union of the boundaries of the interiors of the contours, or equivalently the boundary of the union of the interiors of the contours. In a sense, this is capturing the intuition of “boundary of \( A \)” in the combinatorial sense, but I won’t try to come up with a separate definition to draw a further equivalence. The edges in the union of the traces of the contours are precisely those whose corresponding segments intersect a unit line segment between two adjacent members of \( \mathbb{Z}^2 \) for which exactly one is in \( A \).

End of failed attempts.

Clearly, this field is treacherous. What we were really after is just that any finite subset of \( \mathbb{Z}^2 \) has a finite collection of disjoint contours so that the collection of edges is precisely the set of edges which separate a member and a nonmember in the obvious way. There is no unique determination in any reasonable way. This is proven by induction on the size of the finite subset.

The length of any contour must be even, hence at least 4. Corresponding to every configuration \( \eta \in \{0, 1\}^{\mathbb{Z}^2} \) with at most finitely many 0’s we therefore have the existence of some \( B(\eta) \) a finite collection of disjoint contours whose edges are the divide between 0’s and 1’s. (We are choosing here one of these for each \( \eta \) and denoting it \( B(\eta) \).)

For any finite collection of contours, we will use the absolute value notation to denote the sum of the lengths.

The map \( \eta \mapsto B(\eta) \) which goes from \( \{0,1\}^{\mathbb{Z}^2} \) with at most finitely many 0’s to the space of finite collections of contours is injective.

As a corollary to theorem 3 we have that

**Theorem 5.** For any \( \eta \) with at most finitely many 0’s, every point assigned a 0 is an interior lattice point for at least one of the contours in \( B(\eta) \).

**Proof.** Let \( x \in \mathbb{Z}^2 \) with \( \eta(x) = 0 \). Consider the finite path going straight up until you never see a 0 again. At least one of the contours must be crossed an odd number of times. Thus it switches from inside to outside of this particular
contour at every crossing, hence it changes from inside to outside or outside to inside. But it must end outside because you never see a 0 again so there are no more pieces of the Jordan curve this far out. And then that means it must have started out inside, hence x is in the interior lattice.

3 P. 160 Nonexplosiveness of the MC

Throughout this section, since there will be no occasion to refer to generators since we are dealing with MCs for the moment and not Feller processes, I will use $\Omega, P$ to denote the probability space of an instantiation of the Liggett Probabilistic Construction in Ch. 2 Sec 5 of his 2010 book. (It comes with an implied measurable structure to which we will never refer by symbol. The $i = 1$ and $i = 2$ cases will be discussed separately, so I will use overlapping notation for the discussions, including this and the notation for my own construction which will be $\Omega', P'$.) I will also have the notation from around p.160 in IPS. This overlaps in the use of $c(x)$ but the elements of our state space $Y$ will be denoted by $A, B, \ldots$ and it is not the elements of $S$ that make up our state space, so there is no confusion if I am referring to the Liggett 2010 $c(A) = -q(A, A)$ or the $c(x)$ from IPS. I will carry out my own construction to aid in the understanding of the questions at hand, as was done in the AIMlog with general cts time MCs. As with there, it will be prudent to be sure that I consider the possibility that there is no minimum next jump time originating from the correct site, all default cases, (if the rates are too high and the jump times coalesce, or if they are 0 and the time ends up being equal to infinity.) well-definedness, a.s. issues, correct use of the GMP SMP, the usual conditions on the probability space of my construction, and the increments of the jump times as playing the role of Liggett’s $\tau$s. Also, I will need to make sure that self-transitions are not being counted. Also, I will need to be sure I have done a special case definition for $T_1$ and done my construction in alignment to how $i = 1$ and $i = 2$ cases proceed. We will need to do this once for $i = 1$ and once for $i = 2$. Notice that for both, it will end up being the case once we see that they are legit setups that the unit mass at the empty set is a stationary distribution. Be sure I respect the dichotomy between whether infinity is in the set or not for $i = 1$ and be sure not to count it as a jump when there is a “nothing” transition. I do not always mention when something only holds a.s., especially if it is clearly also a.s. with the exceptional set being measurable (i.e. strong a.s.) but I am aware. (e.g. when a null default case kicks in)

We begin with $i = 1$.

We will assume that $\Omega', P'$ have been chosen so that they admit an independent family of exponentials indexed by $x \in S, A \in Y', l > 0 X_{x,A,l}$ with parameter $c(x)p(x, A)$. As usual, for all $q \geq 0$ let $T_{x,A,q} = \sum_{l=1}^{q} X_{x,A,l}$, and such that none of these exponentials or erlangs are equal except erlangs of order 0 of course, all the exponentials are $> 0$, and all are everywhere finite just as soon as the parameter is $> 0$, $exp(0)$s are everywhere infinite, and $\forall x \in S, t > 0$ we have $max\{q > 0 | T_{x,A,q} \leq t\}$ is summable in $A \in Y$. (This last condition can
be guaranteed because the number of rings coming from site \( x \) of these Poisson clocks has expected value \( c(x)b(x) \) which is finite. \( \) Heuristically, we will need to ignore the transitions that cause no change, and since the \( x \) to \( \{ x \} \) transition always results in no change, its role in my construction is purely cosmetic. The reason this looks nicer, whereas in the general construction in my AIMlog I do not have the transition from a state to itself being represented by a Poisson clock is because Liggett has a notation for \( p \) (The reason I don’t allow \( \infty \) here is for convenience), although sometimes other values will be used when the "usual" definition fails.

Fix initial value \( C_0 \in Y \), and escape/default value \( B_0 \subset S \) (The reason I don’t allow \( \infty \) here is for convenience), although sometimes other values will be used when the "usual" definition fails.

For \( \Phi \) a finite subset of \( Y \), let \( Q_\Phi \) be defined by \( \forall A \in \Phi, B \in Y \) such that \( A \neq B, q_\Phi(A, B) = q_\Phi(A, B) \). \( \forall A \in Y - \Phi, B \in Y \) such that \( A \neq B, q_\Phi(A, B) = \sum_{x \in A \cap S} c(x)b(x)1_{B=A-\{x\}} \). \( \forall A \in Y, q_\Phi(A, A) = -\sum_{B \in Y, B \neq A} q_\Phi(A, B) \). We will define \( Q \) via \( \forall A, B \in Y \) such that \( A \neq B \) we have \( q(A, B) = q_\Phi(A, B) \) and \( q(A, A) = -\sum_{B \neq A} q(A, B) \). The \( Q_\Phi \) are all nonexplosive. (Because on \( A \in \Phi \) the rates coming out of a given \( A \) are bounded and otherwise you are just marching back to some element of \( \Phi \).)

We will now inductively define \( T_j, Z_j \) which are \([0, \infty]\) and \( Y \) valued RVs respectively. These will play the role of the to-be-verified discrete embedded chain and the jump times for \( Q \), for which taking the increments is supposed to "give the Liggett \( \tau \)s". The \( Z_j \)s are being constructed here instead of later (after the continuous time process defined by updates, as I did before in the construction in my AIMlog) partly because unlike before, I need to refer to the \( Z_j \) in each next step to define \( T_{j+1} \). This is because the way the evolution is in terms of changes to the state as opposed to just flat out transitions of the state, and is cumulative as opposed to before even though unrelatedly, the evolutions are still "Markovian". But we will also, because of the same reason actually, need the \( Z_j \) to define the \( A_t \) anyway, and so they could not possibly come after. As is a.s. well-defined (elsewhere use default values 0 and \( B_0 \)), set

\[
T_0 = 0, Z_0 = C_0.
\]

Then set

\[
T_1 = \min\{T_{x,A,1} | x \in Z_0, A \in Y \text{ such that either } x \notin A \text{ or } A \notin Z_0\}.
\]

Then \( \forall \omega' \in \Omega' \) such that \( T_1(\omega') < \infty \) if \( \infty \in Z_0(\omega') \) set

\[
Z_1(\omega') = [(Z_0(\omega') - \{\text{the unique } x \text{ in the pair } (x, A) \in Z_0(\omega') \times Y \text{ that satisfies } T_1(\omega') = T_{x,A,1}(\omega')\}) \cup \text{the unique } A \text{ in the pair } (x, A) \in Z_0(\omega') \times Y \text{ that satisfies } T_1(\omega') = T_{x,A,1}(\omega') \} - \text{the unique } A \text{ in the pair } (x, A) \in Z_0(\omega') \times Y \text{ that satisfies } T_1(\omega') = T_{x,A,1}(\omega') \cap \{\infty\}
\]

and if \( \infty \notin Z_0(\omega') \) set

6
$Z_1(\omega') = (Z_0(\omega') - \{\text{the unique } x \text{ in the pair } (x, A) \in Z_0(\omega') \times Y \text{ that satisfies } T_1(\omega') = T_{x,A,1}(\omega')\}) \cup$

the unique $A$ in the pair $(x, A) \in Z_0(\omega') \times Y$ that satisfies $T_1(\omega') = T_{x,A,1}(\omega')$.

For the rest of the $\omega'$ use the default value $Z_0$ for $Z_1(\omega')$. (Remember, min sometimes contextually means that it is the inf when realized or equal to $\infty$, otherwise it defaults to some value, say 0.)

For $j > 0$

$$T_{j+1} = \min\{T_{x,A,q} > T_j|q > 0, A \in Y, x \in Z_j \text{ such that either } x \notin A \text{ or } A \notin Z_j\}.$$  

Then if $\omega' \in \Omega'$ is such that $T_{j+1}(\omega') < \infty$ and $\infty \notin Z_j(\omega')$ then

$$Z_{j+1}(\omega') = (Z_j(\omega') - \{\text{the unique } x \text{ in the triple } (x, A, q) \in Z_j(\omega') \times Y \times N \text{ that satisfies } T_j(\omega') = T_{x,A,q}(\omega')\}) \cup$$

the unique $A$ in the triple $(x, A, q) \in Z_j(\omega') \times Y \times N$ that satisfies $T_j(\omega') = T_{x,A,q}(\omega')$.

If instead $\infty \in Z_j(\omega')$ then

$$Z_{j+1}(\omega') = \{Z_j(\omega') -$$

$$\{\text{the unique } x \text{ in the triple } (x, A, q) \in Z_j(\omega') \times Y \times N \text{ that satisfies } T_j(\omega') = T_{x,A,q}(\omega')\} \cup$$

the unique $A$ in the triple $(x, A, q) \in Z_j(\omega') \times Y \times N$ that satisfies $T_j(\omega') = T_{x,A,q}(\omega') \cap \{\infty\}$.

If $\omega'$ is such that $T_{j+1}(\omega') = \infty$

Then use $Z_{j+1}(\omega') = Z_j(\omega')$.

Notice that the $T_j$ are all measurable, but could be $\infty$, and all the $Z_j$ are measurable and $Y$-valued.

For $\omega' \in \Omega'$ where the $T_j(\omega')$ are bounded as $j \geq 0$ varies, put $\forall t \geq 0, A_t = B_0$. For other $\omega'$ put $\forall t \geq 0, A_t(\omega') = Z_t(\omega')$ for the unique $j \geq 0$ such that $t \in \{T_j(\omega'), T_{j+1}(\omega')\}$.

$A_t$ is cadlag and measurable.

Notice that here we have done things slightly differently than in the AIMlog construction. $A_t$ is already defaulting on entire strands in time for "bad" $\omega'$, and it is obtained from the $Z_j$ and $T_j$ using the same operation that is used in Liggett probabilistic construction with default value $B_0$. Notice that this construction, as in the Liggett one (remember there’s a default value that is constant in time), is everywhere cadlag. We will also have occasion to refer to $H_1$ which is defined much like $A_t$ except where $T(\omega') := \text{Sup}_j T_j(\omega')$ is finite we shall instead have the (noncadlag, but RC) definition where the default value only takes place for $t \in [T(\omega'), \infty))$.

Fix $\Phi$ a finite subset of $Y$.

We will now inductively define $T_j, Z_j, \Phi$ which are $[0, \infty]$ and $Y$ valued RVs respectively. These will play the role of the to-be-verified discrete embedded
chain and the jump times for \( Q \), for which taking the increments is supposed to "give the Liggett \( \tau \)". The \( Z_{1,\Phi} \)s are being constructed here instead of later (after the continuous time process defined by updates, as I did before in the construction in my AIMlog) partly because unlike before, I need to refer to the \( Z_{j,\Phi} \) in each next step to define \( T_{j+1,\Phi} \). This is because the way the evolution is in terms of changes to the state as opposed to just flat out transitions of the state, and is cumulative as opposed to before even though unrelatedly, the evolutions are still "Markovian". But we will also, because of the same reason actually, need the \( Z_{j,\Phi} \) to define the \( A_{1,\Phi} \) anyway, and so they could not possibly come after. As is a.s. well-defined (elsewhere use default values 0 and \( B_0 \), set

\[
T_{0,\Phi} = 0, Z_{0,\Phi} = C_0.
\]

Then if \( Z_{0,\Phi}(\omega') \in \Phi \)

Then set

\[
T_{1,\Phi}(\omega') = \min \{ T_{x,A,1}(\omega') | x \in Z_{0,\Phi}(\omega'), A \in Y \text{ such that either } x \notin A \text{ or } A \notin Z_{0,\Phi}(\omega') \}.
\]

Given \( \omega' \) such that \( T_{1,\Phi}(\omega') < \infty \) and \( \infty \notin Z_{0,\Phi} \) we define

\[
Z_{1,\Phi}(\omega') = (Z_{0,\Phi}(\omega') - \{ \text{the unique } x \text{ in the pair } (x,A) \in Z_{0,\Phi}(\omega') \times Y \text{ that satisfies } T_{1,\Phi}(\omega') = T_{x,A,1}(\omega') \}
\]

the unique \( A \) in the pair \( (x,A) \in Z_{0,\Phi}(\omega') \times Y \) that satisfies \( T_{1,\Phi}(\omega') = T_{x,A,1}(\omega') \)

and if instead \( \infty \in Z_{0,\Phi} \) then we set

\[
Z_{1,\Phi}(\omega') = (Z_{0,\Phi}(\omega') - \{ \text{the unique } x \text{ in the pair } (x,A) \in Z_{0,\Phi}(\omega') \times Y \text{ that satisfies } T_{1,\Phi}(\omega') = T_{x,A,1}(\omega') \}
\]

the unique \( A \) in the pair \( (x,A) \in Z_{0,\Phi}(\omega') \times Y \) that satisfies \( T_{1,\Phi}(\omega') = T_{x,A,1}(\omega') \) \& \{ \)

Given \( \omega' \) such that \( T_{1,\Phi}(\omega') = \infty \) we define \( Z_{1,\Phi}(\omega') = Z_{0,\Phi}(\omega') \).

If \( Z_{0,\Phi}(\omega') \notin \Phi \) we have:

\[
T_{1,\Phi}(\omega') = \min \{ T_{x,A,1}(\omega') | x \in Z_{0,\Phi}(\omega'), A \in Y \}
\]

If \( \omega' \) is such that \( T_{1,\Phi}(\omega') < \infty \) then we set

\[
Z_{1,\Phi} = (Z_{0,\Phi} - \{ \text{the unique } x \text{ in the pair } (x,A) \in Z_{0,\Phi} \times Y \text{ that satisfies } T_{1,\Phi}(\omega') = T_{x,A,1} \}).
\]

(Notice I do not replace \( x \) by \( A \) but only delete \( x \). But the associated time is still calculated according to all possible transitions, intuitively speaking. This is what happens outside of \( \Phi \)

and if \( \omega' \) is such that \( T_{1,\Phi}(\omega') = \infty \) then we set \( Z_{1,\Phi}(\omega') = Z_{0,\Phi}(\omega') \).

Fix \( j > 0 \)

If \( Z_{j,\Phi}(\omega') \in \Phi \):
we define

\[ T_{j+1,\phi}(\omega') = \min\{T_{x,A,q}(\omega') > T_{j,\phi}(\omega')|q > 0, A \in Y, x \in Z_{j,\phi}(\omega')\} \]

If \( \omega' \) is such that \( T_{j+1,\phi}(\omega') < \infty \) and \( \infty \notin Z_{j,\phi} \) then we set

\[ Z_{j+1,\phi}(\omega') = Z_{j,\phi}(\omega') - \{ \text{the unique } x \text{ in the triple } (x, A, q) \in Z_{j,\phi}(\omega') \times Y \times \mathbb{N} \text{ that satisfies } T_{j,\phi}(\omega') = T_{x,A,q}(\omega') > 0, \text{ the } A \text{ in the triple } (x, A, q) \in Z_{j,\phi}(\omega') \times Y \times \mathbb{N} \text{ that satisfies } T_{j,\phi}(\omega') = 0 \} \]

If instead \( \infty \in Z_{j,\phi} \)

\[ Z_{j+1,\phi}(\omega') = [Z_{j,\phi}(\omega') - \{ \text{the unique } x \text{ in the triple } (x, A, q) \in Z_{j,\phi}(\omega') \times Y \times \mathbb{N} \text{ that satisfies } T_{j,\phi}(\omega') = T_{x,A,q}(\omega') > 0, \text{ the } A \text{ in the triple } (x, A, q) \in Z_{j,\phi}(\omega') \times Y \times \mathbb{N} \text{ that satisfies } T_{j,\phi}(\omega') = 0 \}] \]

If \( \omega' \) is such that \( T_{j+1,\phi}(\omega') = \infty \) then we set \( Z_{j+1,\phi}(\omega') = Z_{j,\phi}(\omega') \).

If \( Z_{j,\phi}(\omega') \notin \Phi \) then:

we define

\[ T_{j+1,\phi}(\omega') = \min\{T_{x,A,q}(\omega') > T_{j,\phi}(\omega')|q > 0, A \in Y, x \in Z_{j,\phi}(\omega')\} \]

If \( \omega' \) is such that \( T_{j+1,\phi}(\omega') < \infty \) then we set

\[ Z_{j+1,\phi}(\omega') = (Z_{j,\phi}(\omega') - \{ \text{the unique } x \in Z_{j,\phi}(\omega') \text{ for which there is } q > 0, A \in Y \text{ such that } T_{j+1,\phi}(\omega') = T_{x,A,q}(\omega') \}) \]

If \( \omega' \) is such that \( T_{j+1,\phi}(\omega') = \infty \) then we set \( Z_{j+1,\phi}(\omega') = Z_{j,\phi}(\omega') \).

The \( T_{j,\phi} \) and \( Z_{j,\phi} \) are measurable, and the latter is \( Y \)-valued.

For \( \omega' \in \Omega' \) where the \( T_{j,\phi}(\omega') \) are bounded as \( j \geq 0 \) varies, put \( \forall t \geq 0, A_{t,\phi} = B_0 \). For other \( \omega' \) put \( \forall t \geq 0, A_{t,\phi}(\omega') = Z_{j,\phi}(\omega') \) for the unique \( j \geq 0 \) such that \( t \in [T_{j,\phi}(\omega'), T_{j+1,\phi}(\omega')] \).

The \( A_{t,\phi} \) are cadlag measurable. Again we define \( H_{t,\phi} \) as the process where the other defaulting scheme is used, and note that \( A_{t,\phi} \) is obtained from the \( Z_{j} \) and \( T_j \) the same way Liggett’s probabilistic construction could get the continuous time chain in terms of the discrete chain and the jump times, with default value \( B_0 \).

Notice that for the processes \( Z_j \) and \( Z_{j,\phi} \) we have that if \( \omega' \) and \( j \geq 0 \) are such that \( T_{j}(\omega') \) or \( T_{j,\phi}(\omega') \) are \( \infty \) respectively, then starting at this \( j \), without even having to worry about null sets thanks to the niceness of the probability space, we have constancy of the process evaluated at this \( \omega' \). Also notice that this one probability space houses all of these processes for varying starting points \( C_0 \), sometimes-used-default-state \( B_0 \), and varying \( \Phi \) or not having a \( \Phi \) subscript.
Notice that we have for all $\Phi$ finite subset of $Y$ the joint distribution of $Z_{0,\Phi}, \ldots, T_{1,\Phi} - T_{0,\Phi}, \ldots$ is equal to the joint distribution of the discrete embedded chain and the $\tau$s for Liggett’s construction for $Q_\Phi$ taken wrt the measure conditioned on starting at $C_0$. Also, the joint distribution of $Z_{0,\Phi}, \ldots, T_{1} - T_{0}, \ldots$ is equal to the joint distribution of the discrete embedded chain and the $\tau$s for Liggett’s construction for $Q$ taken wrt to the measure conditioned on starting at $C_0$. (Here we have the typical convention that $\infty - \infty = \infty$.) Although there is no reason to expect monotonicity in $\Phi$ of the discrete chains, it does hold for the continuous time chains I just constructed: $\Phi_1 \subset \Phi_2 \subset Y$ both finite implies that $\forall t \geq 0, A_{t, \Phi_1} \cap S \subset A_{t, \Phi_2} \cap S$.

If I use default value $B_0$ for both the Liggett construction and here, then $A_{t, \Phi}$ is cadlag and has the same joint distribution as $X_t$ from the Liggett probabilistic construction for $Q_\Phi$ conditioned on starting at $C_0$. If I use default value $B_0$ for both the Liggett construction and here, then $A_t$ is cadlag and has the same joint distribution as $X_t$ from the Liggett probabilistic construction for $Q$ conditioned on starting at $C_0$.

Let $Q_{\Phi,n}$ be defined by $q_{\Phi,n}(A, B) = q_\Phi(A, B)1_{A \in S \leq n}B \in S \leq n$ and $n > \text{the maximum cardinality of any member of } \Phi$. It is obvious how to modify the construction of the process on my $\Omega'$ for $Q_{\Phi,n}$. The analogous facts for this $Q$ matrix and the process I construct that models it all hold. (Measurability, the two ways to make the continuous time process in analogy to $A_t$ and $H_t$, $A_t$ is cadlag and joint measurable, matching in distribution with Liggett construction stuff (if default values are consistently chosen for the continuous time process, and with no caveats for the discrete embedded chain together with the jump times) and also nonexplosiveness) Because this is an approximation for $Q_\Phi$ which is nonexplosive, as $n \to \infty$ we have a.s. convergence on $\Omega'$ of my analogously constructed process to my constructed process for $Q_\Phi$ at a fixed time $t$ if we use either defaulting convention, and $\Omega'$ can be thought of as supporting a coupling of all my constructed processes so far. (For any choice of $C_0, B_0$.)

Notice that we have that the Kolmogorov Forward Equation hypothesis (p.78 theorem 2.38) is satisfied for $Q_{\Phi,n}$. Then the inequalities before (4.11) p. 160 IPS are satisfied for $Q_{\Phi,n}$ in place of $Q$ and upon operating by $P_t$ (the transition function for $Q_{\Phi,n}$) with left argument $C_0 \in Y$ on the left, I can commute with the operation by $Q_{\Phi,n}$. On the RHS there then is $w$ (which is my symbol for the constant called $\omega$ in IPS so as to not cause confusion within this document) times the expectation starting from $C_0$ of the function $A \mapsto |A \cap S|$. On the left there is the derivative wrt $t$ at time $t$.

Now, I use integration factors to see that we have an inequality and then I obtain (4.12) for $Q_{\Phi,n}$. Now, Fatou shows me that I get the same inequality for $Q_\Phi$ in place of $Q_{\Phi,n}$. In this paragraph and the last, all involved processes are nonexplosive so that the Liggett 2010 results that I used are valid, and so that carries over to my construction as the statement that $B_0$ is insignificant.

Define $\forall \omega' \in \Omega', t \geq 0, D_t(\omega') := \cup_{\Phi \subset Y} \lim_{n \to \infty} H_t, \Phi(\omega') \cap S$ and notice that this is a $Y$-valued process a.s. by the above estimate, and where it is not I set it equal to the empty set.

$|H_t(\omega')|$ has to approach infinity as $t \to T(\omega')$ anywhere where this is finite.
(Think about the complementary event that it is \( \leq \) some bound i.o., and then use the fact that condition (c) and (d) of theorem 2.33 in Liggett’s 2010 book are not only equivalent, but the equations within them hold at the same set of \( \omega \) up to a measure 0 set.)

\( H_t \cap S \) and \( D_t \) agree up to and not including time \( T \), at which point \( H_t \) switches to default value (where \( T < \infty \)) and \( D_t \) keeps on going.

Consider \( M_t := |D_t|e^{-wt} \) (I will now use the symbol \( w \) for the \( \omega \) constant appearing in IPS.) This is an \( L^1 \) continuous time forward supermartingale, since the exponential bound similar to (4.12) holds by MCT. Of course when I speak of a similar exponential bound, the \( C^0 \) superscript on the expectation would be removed as the starting point, the expectation is wrt my probability measure which has no dependence on \( C_0 \), but the process is the one for \( C_0, B_0 \). The filtration being used here is (not necessarily RC) the one generated by the Poisson processes associated to each rate up through time \( t \). We use the GMP MP and an in-principle-write-downable function that takes the Poisson processes and tells me the cardinality at time \( t \) of the intersection of the process \( D_t \) with data \( C_0, B_0 \). Here it pays to notice that \( D_t \) varies measurably as \( C \) varies.

I can get a contradiction using the martingale convergence theorem if \( \{ T < \infty \} \) is not null.

Specifically to enact this, for a contradiction assume that \( M \) is such that \( \{ T < M \} \) is not null. I would recognize that if \( (M_t, \mathcal{F}_t) \) is our original supermartingale, then I obtain a discrete time forward \( L^1 \) supermartingale that is \( (M_{T_j \wedge M}, \mathcal{F}_{T_j \wedge M}) \), and we are then guaranteed an a.s. limit that is a.s. finite, and yet this limit must be infinite wherever (a.s.) \( T < M \). (I wedged with \( M \) and reduced to \( T < M \) in order to ensure that the supermartingale is actually defined.)

We have shown that \( T = \infty \) a.s., and this holds for any starting point \( C_0 \) and any default value \( B_0 \). (\( T \) “depends” on \( C_0 \).) Anyway, this shows that \( Q \) is nonexplosive.

Now for \( i = 2 \) we design a similar process on \( \Omega' \) with all sites instead of just \( \infty \) in \( S \cup \{ \infty \} \) being annihilating instead of just \( \infty \) as in \( i = 1 \). We always have the convention that \( \infty - \infty = \infty \) in these sorts of constructions when we look at the increments of the “jump times” and measurability, cadlag when it applies, joint measurability, correct joint distribution of discrete time chain and times, and correctly distribution of continuous time process when default values are consistently chosen all hold. The usual possibility of mistakes listed in the opening paragraph of this section have been checked for. \( \forall \omega' \) the \( H_i(\omega') \cap S \) from \( i = 1 \) contains the one for \( i = 2 \) on \( [0, T(\omega')] \). Therefore, \( \forall \omega' \) we have that \( \{ T_0(\omega'), \ldots \} \cap [0, \infty) \) for \( i = 2 \) is a subset of the one for \( i = 1 \). Thus we also have nonexplosiveness for \( i = 2 \) after applying this analysis for each \( C_0, B_0 \). It should be mentioned of course that the Q matrix for \( i = 2 \) is just defined by \( q(A, B) = q_2(A, B) \) whenever \( A, B \in Y, A \neq B \) and \( q(A, A) = \sum_{B \neq A, B \in Y} q(A, B) \) whenever \( A \in Y \).