1 Induction

Definition 1.0.1. A logical statement is a statement that can either be true or false.

Example 1.0.2. 1. $1 + 1 = 2$. 2. $\sum_{k=1}^{5} 1 = 5$. 3. $\sum_{k=1}^{6} 1 = 100$. 4. All dogs are green.

The method of induction is a way to show that a list of statements is true, where the list can be indexed by the natural numbers, or some consecutive set of whole numbers in general. Let’s state it at only the generality that we used in class today just to keep things as simple as possible, while still having it be general enough to apply to our situation.

Theorem 1.0.3 (Principle of Mathematical Induction (POMI)). \textit{Suppose that $N \in \mathbb{N}$ and for each $n \geq N$ we have $P_n$ is a logical statement. So we have $P_N, P_{N+1}, \ldots$. If we want to see that all the $P_N, P_{N+1}, \ldots$ are true then we may proceed as follows.}

1. Show that $P_N$ is true.
2. Show that if $n \geq N$ and $P_n$ is true, then so $P_{n+1}$ is true as well.

According to the POMI, upon completing the proofs of (1) and (2), it will follow that $\forall n \geq N, P_n$ is true.

Remark 1.0.4. Notice that the statement (2) which is required of you is a conditional. When you try to do (2), you ASSUME that $n \geq N$ and ASSUME that $P_n$ is true. You do not need to prove $P_n$ is true. You prove $P_{n+1}$ is true.

Remark 1.0.5. The POMI is intuitively correct because if you show that the first one in the list, $P_N$, is true, and then show that each one implies the next, then $P_N$ implies $P_{N+1}$ which implies $P_{N+2}$ etc. (1) is often called the base case, and (2) the induction step.

I will provide a non-linear-algebra example of using the POMI to supplement the ones already given at the end of class today. In the next section, I will show the application of the POMI to the linear algebra example requested today.

Theorem 1.0.6. $\forall n \geq 2$ we have $n^2 \geq 2n$.

Proof. The first step is to recognize this as a problem of induction. We set $N = 2$ and we proceed as follows: For each $n \geq 2$ let $P_n$ be the statement that for that particular $n$ we have $n^2 \geq 2n$. It then suffices to see $\forall n \geq NP_n$ (is true). That is the form ripe for application of the POMI.

Base case: To check $P_2$ we observe that $2^2 = 4 = 2 + 2$.

Induction step: We now assume that $n \geq 2$ and that $n^2 \geq 2n$. We add the quantity $2n + 1$ to both sides, leading to $n^2 + 2n + 1 \geq 2n + 2n + 1 = 4n + 1$. But we know from FOILing that the left side of that is just $(n + 1)^2$ and so we learn that $(n + 1)^2 \geq 4n + 1$. But we know that $4n + 1 \geq 2n + 5$ because
\[ n \geq 2 \] and that \[ 2n + 5 \geq 2n + 2 = 2(n + 1). \] Therefore, in summary we have \((n + 1)^2 \geq 2(n + 1)\) that proves \(P_{n+1}\) which completes the induction step.

Then we are done by the POMI.

\[ \square \]

## 2 How to apply it to problem 10 on the homework

First let me set aside how we will use induction as its own claim:

**Theorem 2.0.7.** Fix \(j \in \mathbb{N}\) and \(n \geq j\). Suppose that \(v_1, \ldots, v_n\) are in a vector space \(V\) which is over a field \(F\). Then I claim that

\[
\sum_{k=j}^{n-1} v_k - v_{k+1} = v_j - v_n.
\]

The reason induction is used here to verify a “telescoping sum” kind of equality is that the number of terms being cancelled is finite, but indeterminate, and so we have to be ready for any number of cancellations to be required. So it is most natural to proceed recursively. We noticed in the example of the last section in this document, as well as the example from today in class, that often in induction proofs, one takes the equality you get to assume in the induction step and adds the “missing” amount to both sides. Here we will be employing the same trick.

**Proof.** Here our choice of starting point \(N\) should be \(j\) and our choice of \(P_n\) whenever \(n\) is some number at least the starting point should be \(P_n\) is the statement \(\forall v_1, \ldots, v_n \in V, \sum_{k=j}^{n-1} v_k - v_{k+1} = v_j - v_n.\)” To use POMI, we check the base case first. For \(n = j\) we need to check \(P_j\). To do that, we need to see \(\forall v_1, \ldots, v_j \in V, \sum_{k=j}^{j-1} v_k - v_{k+1} = v_j - v_j.\) Since there are no terms in the range of summation from \(j\) to \(j - 1\), the statement is true.

For the induction step, we are to assume that \(n \geq j\) is given, as well as that \(P_n\) is true, which is to say that

\[
\forall v_1, \ldots, v_n \in V, \sum_{k=j}^{n-1} v_k - v_{k+1} = v_j - v_n. \tag{1}
\]

We then have to prove \(P_{n+1}\), so to do so we should let \(v_1, \ldots, v_{n+1} \in V\) be arbitrary, and try to see that

\[
\sum_{k=j}^{n} v_k - v_{k+1} = v_j - v_{n+1}.
\]

To arrive at the previous display, which is what we desire, we take equation (1) and add the “missing quantity” \(v_n - v_{n+1}\) to both sides. That results in
\[^{(\sum_{k=j}^{n-1} v_k - v_{k+1}) + v_n - v_{n+1}} = v_j - v_n + v_n - v_{n+1}\].

Notice that, as was the point of the “missing quantity” remark, the additional term on the left hand side can be subsumed as the term in the sum corresponding to the first missing term, \(k = n\). So we simply extend the sum to subsume the extra quantity, leading to

\[^{(\sum_{k=j}^{n} v_k - v_{k+1}) = v_j - v_{n+1}}\]

because there is cancellation on the right side of the \(v_n\) terms.

Having proved the telescoping sum identity, it follows then that in the context of problem 10, we can express any of the \(v_j\) via

\[^{v_j - v_n = \sum_{k=j}^{n-1} (v_k - v_{k+1})}\]

which can be manipulated to isolate the \(v_j\) as

\[^{v_j = \sum_{k=j}^{n-1} v_k - v_{k+1} + v_n}\].

This demonstrates that every vector \(v_j\) is in the span of the set \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n, v_n\}\).

But is every linear combination of \(v_j\) also in the span of the set \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n, v_n\}\)? Yes. I will repeat the calculation done in class because it was done too quickly. I will also do it in only the case that we require here for clarity.

Suppose that I am given a linear combination of the \(v_j\), let’s express it as \(\sum_{j=1}^{n} a_j v_j\) where \(a_j \in F\). Then we can make the replacement from what we derived above.

\[^{\sum_{j=1}^{n} a_j v_j = \sum_{j=1}^{n} a_j (\sum_{k=j}^{n-1} v_k - v_{k+1}) + v_n}\].

Notice how the expression on the right in the last display is indeed a linear combination of the objects found in the list \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n, v_n\}\).

Finally, we note that because we have shown that every linear combination of the \(v_j\) is in the span of \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n\}\), and \(V = span(v_1, \ldots, v_n)\) we then have that \(V = span(v_1, \ldots, v_n) \subset \{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n, v_n\} \subset V\) where the last inclusion holds because \(V\) is the entire vector space, and its spans are subspaces. Therefore, we have proven both that \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1}, v_n, v_n\}\) contains, and is contained in \(V\).