I will handle two cases, showing a construction in terms of Poisson Process-like information. One is the setting of Liggett IPS chapter 3 section 3, i.e. the nearest neighbor setting. The other is the case of the contact process, for which there is actually a graphical representation, shown for instance in Liggett 2010 book p. 162. This latter case ended up not getting handled, as I don’t need it yet.

Let’s handle the first case. (So the given data are $S = \mathbb{Z}^1$, $X = \{0,1\}^S$, $c(x,\eta)$ uniformly bounded, satisfying (0.3) of Ch. 3, continuous in $\eta$, nonnegative, with the additional assumptions that they are translation invariant, attractive, and depend only on the coordinates $\eta(x-1), \eta(x), \eta(x+1)$.

Let $t_i \in (0,\infty)$ satisfy $t_i \uparrow \infty$ as $i \to \infty$ with the increasingness being strict. ($i > 0$.) We can say $t_0 = 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which there exist independent exponential RVs of rate $c_{\sigma_1,\sigma_2,\sigma_3}$ for each point $x$ in the site set $S$ and each $l > 0$. (All independent as one big family) That is, we have an independent family of random variables as $x \in S$, $\sigma_1, \sigma_2, \sigma_3 \in \{0,1\}$, $l > 0$ vary: $X_{x\sigma_1,\sigma_2,\sigma_3,l} = \exp(c_{\sigma_1,\sigma_2,\sigma_3})$. The meaning of $\exp(0)$ here is as usual. Let’s say that whenever the parameter is nonzero, we arrange so that the RV is $\mathbb{R}$ valued, and in any case they’re all strictly positive. We then define $T_{x\sigma_1,\sigma_2,\sigma_3,q} = \sum_{l=1}^q X_{x\sigma_1,\sigma_2,\sigma_3,l}$ ($q > 0$, others quantified as usual). We may further assume, by Borel Cantelli and then restriction to an almost sure subspace, that $\forall \omega \in \Omega$ we have for infinitely many $x < 0$ and infinitely many $x > 0$ that $\forall \sigma_1, \sigma_2, \sigma_3, qT_{x\sigma_1,\sigma_2,\sigma_3,q}(\omega) \notin [t_i, t_{i+1}]$. Notice that which ones are among the guaranteed infinitely many such $x$ depends on both $\omega$ and $i$ but not on $q$. We may also assume by restriction again that $\forall \omega \in \Omega$ any two of the RVs named above have different values at $\omega$, and none of them are equal to any of the $t_i$. In addition, we require that $\forall \omega \in \Omega, x \in S, \sigma_1, \sigma_2, \sigma_3T_{x\sigma_1,\sigma_2,\sigma_3,q} \to \infty$ as $q \to \infty$. (“We may assume” here means that the construction below will only apply when the assumptions here are met, which is possible, but I use this phrase because we only care that there is some reasonable construction that directly relates the rates to the Feller Process.)

Now fix $i \geq 0$, $\omega \in \Omega$ and obtain the two-sidedly infinite collection of sites $x$ as guaranteed by the above, say that they are $x_j$ (strictly increasingly indexed) for $j \in \mathbb{Z}$ with $x_j \to \infty$ as $j \to \infty$ and $x_j \to -\infty$ as $j \to \infty$. Now fix $j \in \mathbb{Z}$ and let $(s_k)_{1 \leq k \leq N}$ be the strictly increasing rearrangement of the values $T_{x\sigma_1,\sigma_2,\sigma_3,q}(\omega)$, as $x : x_j < x < x_{j+1}$, $\sigma_1, \sigma_2, \sigma_3 \in \{0,1\}$, $q > 0$ vary, that lie in $(t_i, t_{i+1})$. Unbind $j$, $i$ now. Then fix $\eta_0 \in X$. First we define $\eta_\omega(\omega)$ on $[t_0 = 0, t_1]$, so we apply the above to $i = 1$. Look at the coordinates between $x_j$ and $x_{j+1}$ and keep them constant except that we update them CADLAGly at every $s_k$ for which the unique $\sigma_1, \sigma_2, \sigma_3, x$ data agrees with the pre-existing configuration, and in this case the update we use is to flip at site $x$. We do this for every $j \in \mathbb{Z}$. Then we take $\eta_\omega(\omega)$ as the starting point and do the same thing again for $[t_1, t_2]$, etc. We prove measurability of $\eta_\omega(x)$ for fixed $\eta_0$ first for $t \in [t_0, t_1]$ and all $x$ and then we move to the next time interval. These are done by writing as a union of where the nearest markers $x_j$ are to either side of $x$ and then unioning over the number of Erlang clock rings, and then recognizing that all that matters.
is the relative ordering of all the clock rings of the varying Erlang clocks, and then since this is a countable collection and some of them lead to the given configuration, and each of them is described as an inequality involving random variables and the $t_i$, then we’re done. Then we recognize that this means $\eta_t$ is measurable into $X$ with Borel structure. This lets us move to the next time interval, because the above argument actually shows joint measurability with the dependence on starting point as well, as would be required. In this proof of measurability, we have the trivial observations for the piece where $x$ itself turns out to be a marker for the point $\omega$ and the iteration $i$ we are on, and of course some contradicting conditions lead to the empty set but that’s measurable too. Every site has a CADLAG path, so the whole thing is CADLAG, and thus a random variable with values in the CADLAG path space for $X$. Hence we can push forward the probability measure and call the result $P^\omega$, and thusly we get a family of probability measures on the CADLAG path space. We want to show that this, together with the canonical RC filtration, constitutes a Feller Process.

There is a natural question I would be willing to ignore, if not for the fact that it will actually be used in the proof. Suppose that I repeat the construction above for a different sequence of $t_i$, and thus I get a different collection for each $i$ and $\omega$ of $x$ values. In fact, we may even assume that I do not take every one of them, but just some infinite subcollection of them. Then I get the same process, up to equality, so measurability is no problem. This enables me to basically recognize the independence of construction on the choices, but importantly it allows me instead of seeing time and sites partitioned according to many small bricks, if I have a finite time question such as some of the verifications that will occur in the Markov Property check, I can envelop the entire domain of interest into one big time by site block. These facts can be seen by just noticing that all I do is update sites, and that doesn’t depend on where I “think of” the boundary as being. (Everything else must be kept the same, such as the exponential/Erlang RVs themselves, etc.) Notice that I cannot take the block independent of $\omega$, but what I can do is for a fixed block, I can choose a larger block at every $\omega$.

The Feller property follows from the fact that if we use the above to enclose a finite dimensional function’s coordinates that matter for all $\omega$ at the same time, say we choose to do so by the closest possible sites. Then where these sites are are geometric random variables on the original probability space. But then conditional on fixing these “walls” the conditional expectation in undergrad sense of that function evaluated at $\eta_t$ is then a finite dimensional function, hence continuous. So the sum is then continuous by standard convergence stuff. But then the same is true on the CADLAG path space.

To check the Markov Property (MP) first we see that if we take the expectation of a product of finite dimensional functions precomposed with $\eta_t$ for some fixed time set $\{t_i\}$ then we can extend to be an infinite list of the type above, and view the construction as having been done for that list. However, when I find the $x_j$ I will want to do so in a way that includes the coordinates that all the pieces care about, and so I go out some preset distance from the
totality of these coordinates, and condition starting from there. Is it this one? Or is it the next one and not the last one? Since there are infinitely many, it doesn’t matter that I started asking with one that is a fixed finite distance away. Now, I partition the other way: I condition: on which time slices are the ones blocking the bars from moving in closer. Eventually I break down into Poisson Process raindrops into each of the time slices, and then I split over these by independence, and then I recombine via independence again, after splitting off the wall-pusher-outers by realizing that the conditional probability of a path segment conditioned on where the closest possible wall for a time slice is is the same as just the probability of that path, and voila. (done after using the insignificance of the choice of which bars to take.) This shows that the obvious formula for the expectation of such a finite time finite site product is actually the iterated expectation that we expect it to be.

Upgrading to MP is routine. Thus our construction gives a Feller Process. By simple DCT arguments involving estimates on the probability for a double occurrence in a small interval and for single occurrences, we see that the form of the generator is correct on finite dimensional indicators. Thus on all finite dimensional functions. Thus since it is bounded, on all continuous functions. Thus, the Feller process so obtained is the same as the one obtained in the usual way from the rates. ACTUALLY THIS WAS WRONG. You need to check it on a core, not just a dense subset. Instead, set the rates to be 0 outside of a finite subset of $Z^1$ and use BCT and trotter approximation to see directly that the semigroups are equal.

To verify the $\epsilon = 0$ case of Liggett’s Ch. 3 Sec 3 argument for theorem 3.13, one goes back to the probability space above, as opposed to the CADLAG path space, and demonstrates that, for the construction starting with any $\eta_0$, modulo a $P$-null set in $\omega$, the limit at $\omega$ is the constant 1 vector iff there is some time at which the sample path adopts a state where two 1s are adjacent, and the constant 0 vector otherwise. Notice that the notion of a.s. limit here in $X$ is fine wrt measurability issues, despite the continuous parameter, and as is the fact that I cased out based on if for any $t$ there are adjacent 1s, in both cases by CADLAGness. Notice also that I have made extensive use of waiting for Poisson-type events to happen as we are in that setting. This a.s. convergence is enough to show that if $\nu$ is an invariant measure on $X$ in the usual sense for the rates, then since the probability that I start at $\eta_0$ and end up with 2 adjacent 1s depends measurably on $\eta_0$ then I get that $\nu$ is a convex combination of the point masses at the 0 and 1 vectors. If $c_{000} > 0$ then the weight in the convex combination assigned to $\delta_0$ is 0. Otherwise, $\delta_0$ is the lower invariant measure. Anyway, $\delta_1$ is the upper invariant measure. Analogous for the other way. Here the 0 and 1 denote vectors. Thus the $\epsilon = 0$ case is resolved.

This can be generalized to the case where $\{0, 1\}$ is replaced by a finite set $W$ and $Z^1$ is replaced by any connected, non-multiple-edge, non-loop, at most countable, locally finite graph with critical site percolation probability $> 0$. The necessary changes are that we must give an a priori enumeration of the vertices around each vertex, and we must replace the boundaries discussed above by the graph theoretic boundary of a finite subset when we try to split into islands.
Of course, it is also necessary to introduce many more Poisson RVs in order to track all the possible transitions. In this case, it is also necessary to assume that the total transition rates at each site is uniformly bounded as I vary the site.

Under the stronger hypothesis of bounded degree, we have the contact process with its own special graphical description, and it arises as a corollary of the work done here.