1 Solution to Problem 11 of Section 1.3

Please note the email in which I discussed a mistake I made. The mistake did not affect the solution, but I also went over the material fairly quickly, and doing it again as a set of notes serves to show how the mistake does not affect the problem. So here it is.

Problem statement clarified (a student pointed out that the book is ambiguous about whether there is one $W$ for all the $n$ or, whether there is one $W$ for each $n$. The fact that I fix $n$ at the outset means that there is one possibly different $W$ for each $n$): Fix an $n \geq 1$. Let $W$ be defined as the set of polynomials $f(x)$ within $P(F)$ for which the degree of $f(x)$ is either $n$ or $-1$ (That would mean $f(x) = 0$.) Prove or disprove that $W$ is a vector subspace of $P(F)$.

Disproof: Consider the case of $n = 2$ for example. Let $f(x) = x^2 - 1$ and let $g(x) = -x^2$, so that $f(x) + g(x) = -1$ and is thus neither 0 nor degree 2 = $n$. If $n$ is instead general, then modify this by letting $f(x) = x^n - 1$ and $g(x) = -x^n$ and the same problem ensues. Thusly, we have shown an example of one $f(x)$ and one $g(x)$ for which we have that $f(x)$ and $g(x)$ are degree $n$, and thus in $W$, whereas their sum is equal to $-1$, which is neither of degree $-1$ nor of degree $n$.

Remember how logic works. If $P$ and $Q$ are logical statements, then $\neg(P \lor Q) = (\neg P) \land (\neg Q)$. Similarly, $\neg \forall$ is the same as $\exists \neg$. The order of logical operations in mathematical reasoning is extremely important for this reason. To prove that unicorns exist, I need only to find one. To prove $\neg (\text{unicorns exist})$, I have to show that in every spec of air there are no unicorns. That is, I must show that $\forall$ space, that space does not contain a unicorn. See how “there exists” became a “for all” after negation?

Regarding the comment with $P_n(F)$ being similar to $F^{n+1}$ consider the correspondence of $x^{i-1}$ with $e_i$ where $e_i$ is the unit vector in $F^{n+1}$ with a one in the $i$th coordinate. Notice that different powers of $x$ do not combine when I add vectors, just as different components do not combine when I compute sums in $F^{n+1}$. We will talk about this more later, when we speak of bases.