

AN EXPLICIT UPDATE SCHEME FOR INVERSE PARAMETER AND INTERFACE ESTIMATION OF PIECEWISE CONSTANT, DISCONTINUOUS COEFFICIENTS IN LINEAR ELLIPTIC PDES *

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Abstract. We introduce a general and efficient method to recover piecewise constant coefficients occurring in elliptic partial differential equations as well as the interface where these coefficients have jump discontinuities. For this purpose, we use an output least squares approach with level set and augmented Lagrangian methods. Our formulation incorporates the inherent nature of the piecewise constant coefficients, which eliminates the need for a complicated non-linear solve at every iteration. Instead, we obtain an explicit update formula and therefore vastly speed up computation. We employ our approach to the example problems of Poisson’s equation and linear elasticity and provide the combination of simultaneously recovering coefficients and interface.

Key words. Inverse problems, elliptic problems, inverse parameter estimation, inverse geometric problems, Poisson’s equation, linear elasticity, augmented Lagrangian method, level set method

AMS subject classifications. 65N21, 65N20, 65K10, 49M05

1. Introduction. The identification of coefficients occurring in elliptic partial differential equations is a problem that occurs in several different fields, including solid and fluid mechanics, image processing and many more (see e.g. [47]). In the case of medical imaging, inverse parameter estimation can potentially be used to determine properties and location of different tissue types while using minimally invasive technologies (see [45] and references therein) instead of dissecting the patient. Classifying the elastic properties of tissue and locating abnormalities can help to identify and pinpoint cancerous growth [46]. Further, the elastic properties of bones, muscles and tissue in contact can be approximated as piecewise constant but their actual values may vary from patient to patient. Knowing their exact characteristics can be used to reset, adjust and fine tune simulations [56]. This allows for better understanding of the biomechanical configuration at hand and therefore better, individualized treatment for every patient.

In [32], Ito and Kunisch suggested combining an output least squares and equation error formulation with the augmented Lagrangian method to solve inverse parameter estimation problems and that approach has since become very successful, see e.g. [16, 17, 19, 26, 40, 70]. Usually, these methods discretize the coefficients over the entire domain and use a total variation regularization to achieve an essentially piecewise constant solution for the coefficients. This, and the homogeneous structure of the

*This work was supported by the National Science Foundation under grants DMS-0502315, DMS-0652427, CCF-0830554, by UC Lab Fees Research Grant DOE 09-LR-04-116741-BERA, and the Office of Naval Research under grants ONR N000140310071, and ONR N000141010730.

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[§]This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-0707424.

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coefficients in many applications, e.g. geophysical sciences [4, 39], inverse scattering [43] and medical imaging as mentioned before, motivates to include the piecewise constant nature of the coefficients into the approach to these problems.

In addition to estimating the coefficients of a PDE, the geometry of the domain of interest is important for many applications, such as the locations of cavities within a material or an optimal shape design under certain exterior conditions. For this so-called inverse geometric problem, where the unknown is a geometric shape, many different approaches have been proposed, e.g. [28, 29, 30, 38, 39]. In [57], Santosa suggested the use of the level set method, developed by Osher and Sethian [52], for inverse obstacle problems. The implicit representation of an interface as the zero level set of a function allows for natural handling of topology changes, such as splitting and merging. Therefore, the level set is capable of evolving towards a solution from almost any initial shape, requiring little or no a-priori knowledge. This versatility is highly desirable for solving shape optimization and shape reconstruction problems since the topology of the solution is usually unknown, cf. [10, 11, 12, 13, 14, 23, 33, 51, 53, 54, 55, 62, 65, 66].

Much of the previous work has been done on the model inverse problem arising from Poisson's equation

$$\begin{cases} -\nabla \cdot \beta \nabla u = f & \text{in } \Omega \\ u = g_1 & \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial n} = g_2 & \text{on } \partial\Omega_N, \end{cases}$$

with applications ranging from electrical impedance tomography [9, 20, 62] and DC resistivity [60] to ground water and oil reservoir investigations [36, 37]. Some authors, e.g. [18, 22, 48, 64], have combined the two aforementioned approaches for the case of Poisson's equation. The goal of these papers is to recover the coefficients as well as the unknown interface.

There has been some work to extend these or similar techniques to the related elliptic inverse problem originating in linear elasticity

$$\begin{cases} -\nabla \cdot \sigma(u) = f & \text{in } \Omega \\ u = \bar{u} & \text{on } \partial\Omega_D \\ \sigma \cdot n = \bar{t} & \text{on } \partial\Omega_N, \end{cases}$$

where

$$\begin{aligned} \sigma(u) &= 2\mu\epsilon(u) + \lambda \operatorname{tr}(\epsilon(u)) I \\ \epsilon(u) &= \frac{1}{2}(\nabla u + \nabla u^T), \end{aligned}$$

with applications in material property determination and inclusion detection [41, 58] as well as design optimization [1, 2, 8, 15, 59] and medical imaging [31, 35]. These works have been either to estimate the Lamé parameters [24, 25, 34, 49] or to solve the inverse geometric problem [3, 7].

In the present paper, we also combine the augmented Lagrangian approach with the level set method. As opposed to the aforementioned work of [18], we do not use the whole finite element space to approximate the coefficients in each element combined with a total variation regularization. That approach requires an additional non-linear

solve for the coefficients at every grid point at every iteration. The versatility of allowing the coefficients to spatially vary is typically dominated by the TV norm, which leads to smoothing of the solution and to essentially piecewise constant results.

Our approach incorporates the information about the nature of the piecewise constant coefficients directly into the method, which allows us to avoid the additional solve corresponding to the coefficients and instead obtain an explicit, and therefore very efficient, update at each iteration. The apparent limitation to truly piecewise constant coefficients is justified in many applications, where the materials are essentially homogeneous. Other authors have successfully included the assumption of piecewise constant coefficients into their approaches [21, 22, 48, 64]. However, our method uses a different objective functional and is more efficient because we use only one linear solve per iteration, all other variables are updated explicitly. For simplicity, we limit this paper to the example problems of Poisson's equation and linear elasticity, but the methodology is rather general and can be extended to other inverse problems constrained by linear elliptic PDEs.

2. Problem Formulation. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, with a smooth or piecewise smooth boundary $\partial\Omega$. We model two elliptic inverse problems: Poisson's equation as well as linear elasticity. Due to the ill-posedness of both inverse problems, we use output-least-squares with an observation $u_0 \in L^2(\Omega)$ of the solution to recover the unknown coefficients occurring in the respective partial differential equation. We allow the coefficients to have jump discontinuities across some unknown interface Γ in Ω and recover Γ as well. This interface Γ , which separates the domain into two disjoint open sets Ω_1 and Ω_2 , is assumed to be (piecewise) smooth and is represented implicitly by a level set function ϕ (for more details on the level set method we refer to [50]). The subsets Ω_1 and Ω_2 correspond to the regions of positive and negative function values of ϕ , see Figure 1:

$$\begin{aligned}\Omega_1 &= \{x \in \Omega \mid \phi(x) < 0\}, \\ \Omega_2 &= \{x \in \Omega \mid \phi(x) > 0\}, \\ \Gamma &= \{x \in \Omega \mid \phi(x) = 0\} = \partial\Omega_1 \setminus \partial\Omega = \partial\Omega_2 \setminus \partial\Omega.\end{aligned}$$

(The limitation to two regions is for simplicity only, a generalization to arbitrary numbers of subregions is easily obtainable via multiple level sets, see e.g. [18, 63, 68].) We assume the coefficients to be constant in each Ω_i and denote the restriction of a value in one region by a subscript i .

2.1. Poisson's equation. We employ Poisson's equation as a first case to show the simplicity of our approach:

$$\begin{cases} -\nabla \cdot \beta \nabla u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_N, \end{cases} \quad (1)$$

with piecewise constant coefficient

$$\beta = \begin{cases} \beta_1 & \text{in } \Omega_1 \\ \beta_2 & \text{in } \Omega_2. \end{cases} \quad (2)$$

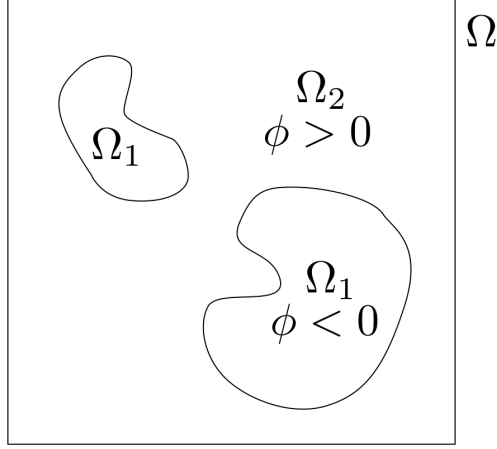


FIG. 1. *Problem setting: Two materials, represented by Ω_1 and Ω_2 , with different, unknown properties are in contact. The interface between them, where the parameters have a jump discontinuity, is unknown as well.*

We solve the constrained minimization problem

$$\min_{u \in H^1(\Omega), \Gamma} \int_{\Omega} |u - u_0|^2 dx + w|\Gamma| \quad \text{such that } u \text{ is a solution to (1),} \quad (3)$$

where $|\Gamma| = \mathcal{H}^{d-1}(\Gamma)$ denotes the $(d-1)$ -dimensional Hausdorff measure of Γ and $w > 0$ is a weighting parameter. The corresponding augmented Lagrangian functional for any $r > 0$ is defined as

$$\begin{aligned} \mathcal{L}_A(u, v, \beta_1, \beta_2, \Gamma; r, w) &= \frac{1}{2} \int_{\Omega} |u - u_0|^2 dx + \int_{\Omega} (\nabla v \cdot \beta \nabla u - f v) dx \\ &\quad + \frac{r}{2} \int_{\Omega} |\nabla \cdot \beta \nabla u + f|^2 dx + w|\Gamma|, \end{aligned} \quad (4)$$

where we try to enforce the equation error constraint in the $H^{-1}(\Omega)$ -norm. We find its saddle point by solving the first order optimality conditions:

$$\frac{\partial \mathcal{L}_A}{\partial u} = u - u_0 - \nabla \cdot \beta \nabla v + r \nabla \cdot \beta \nabla (\nabla \cdot \beta \nabla u + f) = 0, \quad (5)$$

$$\frac{\partial \mathcal{L}_A}{\partial \beta_i} = \int_{\Omega_i} \nabla v \cdot \nabla u dx + r \int_{\Omega_i} (\nabla \cdot \beta_i \nabla u + f) \Delta u dx = 0 \quad \text{for } i = 1, 2. \quad (6)$$

For the derivative with respect to the geometry, we use some classical results about shape derivatives, as given in Appendix A. In order to apply these lemmas, we split the integral into the subsets Ω_i , yielding

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial \Omega}[\theta] &= \sum_{i=1}^2 \int_{\partial \Omega_i \setminus \partial \Omega} \left(\frac{1}{2} |u - u_0|^2 + \nabla u \cdot \beta_i \nabla v - f v \right. \\ &\quad \left. + \frac{r}{2} |\nabla \cdot \beta_i \nabla u + f|^2 \right) \theta(x) \cdot n_i(x) ds + w \int_{\partial \Omega_i \setminus \partial \Omega} \kappa_i(x) \theta(x) \cdot n_i(x) ds \quad (7) \\ &= 0, \end{aligned}$$

where n_i is the outside normal to Ω_i , and $\kappa_i = \nabla \cdot n_i$ is the mean curvature of $\partial\Omega_i$. Since $n_2 = -n_1$ on $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ and thus also $\kappa_2 = -\kappa_1$, we are left with the following equation that needs to vanish on the interface

$$V = \nabla u \cdot (\beta_1 - \beta_2) \nabla v + w \kappa. \quad (8)$$

With A representing the operator $(-\nabla \cdot \beta \nabla)$ and since $A^* = A$ for this problem, we can rewrite (5) as

$$(rA^2 + I)u = u_0 - Av + rAf, \quad (9)$$

with the boundary conditions specified in (1).

2.2. Linear elasticity. We mainly focus on the identification of unknown interfaces and material parameters in the more complicated case of linear elasticity. The governing equations are given by

$$\begin{cases} -\nabla \cdot \sigma(u) = f & \text{in } \Omega \\ u = \bar{u} & \text{on } \partial\Omega_{\text{D}} \\ \sigma \cdot n = 0 & \text{on } \partial\Omega_{\text{N}}. \end{cases} \quad (10)$$

The displacement u and the associated stress $\sigma(u)$ are related by Hooke's law

$$\sigma(u) = C : \epsilon(u) = 2\mu\epsilon(u) + \lambda \operatorname{tr}(\epsilon(u)) I, \quad (11)$$

where C is the fourth order elasticity tensor and the strain is given by

$$\epsilon(u) = \frac{\nabla u + \nabla u^{\text{T}}}{2} \quad (12)$$

The Lamé coefficients

$$\mu = \frac{E}{2(1+\nu)}, \quad (13)$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad (14)$$

depend on Young's modulus E and Poisson's ratio ν . We consider a piecewise homogeneous material, thus μ and λ are piecewise constant and are defined analogously to (2). In our approach we solve for μ and λ but use (13) and (14) to recover the physically meaningful material parameters E and ν .

We have a similar constrained minimization problem,

$$\min_{u \in H^1(\Omega), \Gamma} \int_{\Omega} |u - u_0|^2 dx + w|\Gamma| \quad \text{such that } u \text{ is a solution to (10)}. \quad (15)$$

Again, we define the corresponding augmented Lagrangian functional for any $r > 0$ as

$$\begin{aligned} \mathcal{L}_A(u, v, \mu_1, \lambda_1, \mu_2, \lambda_2, \Gamma; r, w) &= \frac{1}{2} \int_{\Omega} |u - u_0|^2 dx \\ &+ \int_{\Omega} ((2\mu\epsilon(u) + \lambda \operatorname{tr}(\epsilon(u)) I) : \epsilon(v) - fv) dx \\ &+ \frac{r}{2} \int_{\Omega} |\nabla \cdot \sigma(u) + f|^2 dx + w|\Gamma|, \end{aligned} \quad (16)$$

and compute its optimality conditions

$$\frac{\partial \mathcal{L}_A}{\partial u} = u - u_0 - \nabla \cdot \sigma(v) + r \nabla \cdot \sigma(\nabla \cdot \sigma(u) + f) = 0 \quad (17)$$

and for $i = 1, 2$

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial \mu_i} &= 2 \int_{\Omega_i} \epsilon(u) : \epsilon(v) dx \\ &\quad + 2r \left(\int_{\Omega_i} \nabla \cdot (2\mu_i \epsilon(u) + \lambda_i \text{tr}(\epsilon(u))I) \cdot (\nabla \cdot \epsilon(u)) dx \right. \\ &\quad \left. + \int_{\Omega_i} f \cdot (\nabla \cdot \epsilon(u)) dx \right) \\ &= 0 \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial \lambda_i} &= \int_{\Omega_i} \text{tr}(\epsilon(u))I : \epsilon(v) dx \\ &\quad + r \left(\int_{\Omega_i} \nabla \cdot (2\mu_i \epsilon(u) + \lambda_i \text{tr}(\epsilon(u))I) \cdot (\nabla \cdot \text{tr}(\epsilon(u))I) dx \right. \\ &\quad \left. + \int_{\Omega_i} f \cdot \text{tr}(\epsilon(u))I dx \right) \\ &= 0. \end{aligned} \quad (19)$$

Using the same argument and notation as before, the shape derivative in this case is given by

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial \Omega} [\theta] &= \sum_{i=1}^2 \int_{\partial \Omega_i \setminus \partial \Omega} \left(\frac{1}{2} |u - u_0|^2 + \epsilon(u) : C_i : \epsilon(v) - f v \right. \\ &\quad \left. + \frac{r}{2} |\nabla \cdot \sigma_i(u) + f|^2 \right) \theta(x) \cdot n_i(x) ds + w \int_{\partial \Omega_i \setminus \partial \Omega} \kappa_i(x) \theta(x) \cdot n_i(x) ds \quad (20) \\ &= 0 \end{aligned}$$

yielding

$$V = \epsilon(u) : (C_1 - C_2) : \epsilon(v) + w \kappa, \quad (21)$$

which needs to vanish on the interface.

Again, using the corresponding linear operator $A = \nabla \cdot \sigma(\cdot)$, we obtain u by solving

$$(rA^2 + I)u = u_0 - Av + rAf \quad (22)$$

with boundary conditions of the original problem (10).

For a more rigorous analysis of augmented Lagrangian methods for identifying discontinuous parameters in elliptic systems, we refer the interested reader to [19, 26]. For a detailed analysis of convergence of level set methods for elliptic inverse problems as well a more general functional-analytic framework of level set methods and shape reconstruction, we refer to [10, 11, 13].

3. Optimization Algorithm. In order to solve the equations derived in the previous section, we use an alternating optimization algorithm. We denote the iterations by k and discretize all operators and functions in an appropriate manner.

3.1. Coefficient Updates. For the coefficients, we use the optimality conditions to derive an explicit update rule. In the case of Poisson's equation, we assume $f \neq 0$ and solve (6) at iteration $k + 1$ for β_i^{k+1} . Since β_i is constant on Ω_i , we can substitute $\Delta u = -\frac{f}{\beta_i^k}$ on the right hand side, yielding

$$\beta_i^{k+1} = \beta_i^k - \frac{\int_{\Omega_i} \nabla v \cdot \nabla u dx}{r/(\beta_i^k)^2 \int_{\Omega_i} f^2 dx} \quad \text{for } i = 1, 2. \quad (23)$$

Similarly, for linear elasticity we solve (18) and (19) for the coefficients μ_i and λ_i and then substitute in their respective values from the previous iteration:

$$\mu_i^{k+1} = \mu_i^k - \frac{\int_{\Omega_i} \epsilon(u) : \epsilon(v) dx}{r \int_{\Omega_i} |\nabla \cdot \epsilon(u)|^2 dx}, \quad (24)$$

$$\lambda_i^{k+1} = \lambda_i^k - \frac{\int_{\Omega_i} \text{tr}(\epsilon(u)) I : \epsilon(v) dx}{r \int_{\Omega_i} |\nabla \cdot \text{tr}(\epsilon(u)) I|^2 dx}. \quad (25)$$

3.2. Interface Update. As mentioned in Section 2, we represent the interface Γ implicitly by the zero contour of a level set function ϕ . A natural choice for ϕ is the signed distance to the interface

$$\phi(x) = \begin{cases} \text{distance}(x, \Gamma) & x \in \text{interior of } \Gamma \\ -\text{distance}(x, \Gamma) & x \in \text{exterior of } \Gamma. \end{cases} \quad (26)$$

The evolution of ϕ throughout the whole domain through an artificial time can be modeled by a Hamilton-Jacobi equation of the form

$$\frac{\partial \phi}{\partial t} + V \delta(\phi) = 0, \quad (27)$$

with an initial shape

$$\phi(x, 0) = \phi_0(x). \quad (28)$$

The speed V is given by Equations (8) and (21) for Poisson's equation and linear elasticity respectively. The level set evolution reaches its steady state once the optimality condition is fulfilled.

We use a semi-implicit finite difference scheme based on the approach by Chan and Vese (see [68]) to discretize the level set evolution (27), and approximate the interface using the following C^∞ regularization of the delta-function

$$\delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}. \quad (29)$$

Since it is known ([5, 27]) that the solution to (27) is not generally a signed distance function and in order to keep ϕ from getting too flat or too steep, we reinitialize it from time to time as a signed distance function by a fast-sweeping step following [69]. One has to be careful not to reinitialize too often. Our inverse problems are

ill-posed and therefore, the evolution is very slow and it might take a long time for a nodal value to change sign. By reinitializing too often, these small changes could be revoked. In our method, we reinitialize either when a certain percentage of nodal values of ϕ have changed signs or after a fixed number of iterations have passed since the last reinitialization. A different strategy is to reinitialize after the L_2 norm of ϕ has changed more than a given percentage.

3.3. Algorithm. We now outline our algorithm. First, we initialize ϕ^0 with the initial guess for the interface, the coefficients (β_i^0 or μ_i^0 , λ_i^0 , ($i = 1, 2$) respectively) with positive values and v^0 as a zero-vector. We also choose a fixed number of iterations *reinit_number* and a percentage *reinit_ratio* for the level set reinitialization condition, as well as initialize the helper variables *number_sign_changes* and *iterations_since_reinit* as zero.

```

Initialization
for all  $k$  do
  Solve  $(rA^2 + I)u^{k+1} = u_0 - Av^k + rAf$ 
   $v^{k+1} \leftarrow v^k + r(Au^{k+1} - f)$  // explicit update of the Lagrange multiplier
  for all  $i = 1$  to  $2$  do // explicit update of the coefficients
    // in case of Poisson's equation
     $\beta_i^{k+1} \leftarrow \beta_i^k - \frac{\int_{\Omega_i} \nabla v^{k+1} \cdot \nabla u^{k+1}}{r / (\beta_i^k)^2 \int_{\Omega_i} f^2}$ 
    or // in case of linear elasticity
     $\mu_i^{k+1} \leftarrow \mu_i^k - \frac{\int_{\Omega_i} \epsilon(u^{k+1}) : \epsilon(v^{k+1})}{r \int_{\Omega_i} |\nabla \cdot \epsilon(u^{k+1})|^2}$ 
     $\lambda_i^{k+1} \leftarrow \lambda_i^k - \frac{\int_{\Omega_i} \text{tr}(\epsilon(u^{k+1})) I : \epsilon(v^{k+1})}{r \int_{\Omega_i} |\nabla \cdot \text{tr}(\epsilon(u^{k+1})) I|^2}$ 
  end for
  Advect( $\phi^k, \phi^{k+1}$ ) // as described in Section 3.2
  for all  $i = 1$  to number_nodes do // check  $\phi$  for a sign changes
    if  $\phi_i^k \cdot \phi_i^{k+1} < 0$  then
      number_sign_changes  $\leftarrow$  number_sign_changes + 1
    end if
  end for
  if iterations_since_reinit > reinit_number
  or number_sign_changes/number_nodes > reinit_ratio then
    Fast_sweep( $\phi^{k+1}$ ) // reinitialization as a signed distance function
    number_sign_changes  $\leftarrow$  0
    iterations_since_reinit  $\leftarrow$  0
  end if
end for

```

Remarks:

(i) The update order for the different variables is not important. Given an order, it is most efficient to use the most recent values on hand to update the remaining variables.

(ii) Note that $(rA^2 + I)$ never has to be built explicitly, only its application is needed, which makes computations efficient. The matrix $(rA^2 + I)$ is obviously symmetric and positive definite, therefore we can use a (preconditioned) conjugate gradient method or similar to solve the equation to obtain u . We need only one solve

per iteration since both the Lagrange multiplier v and the coefficients are updated explicitly.

(iii) We discretize the equations using the finite element method (FEM). We embed the interface by piecewise linear approximation into each element. We then use the true area weights to assemble the FEM stiffness matrix A and evaluate the integration exactly on each partition.

4. Numerical Examples. For all examples we use a regular triangle grid of the domain $\Omega = [0, 1]^2$ with uniform spacing $\Delta x = \Delta y = h = 1/256$. The PDE solves for u are completed using FEM. The time step of the level set evolution is fixed to $\Delta t = h$ and coincides with a minimizing step of the Lagrangian functional \mathcal{L}_A . For the approximation of the interface in (29), we set $\epsilon = h$. We reinitialize ϕ either when 10% of the vertices have changed sign or 100 iterations have passed since the last reinitialization. The choice of r highly affects the speed of convergence and the convergence is typically very fast in the beginning and then slows down. For each example, we choose an appropriate r that leads to optimal convergence behavior.

4.1. Poisson's Equation. We demonstrate our algorithm for the case of Poisson's equation using an example where Ω_1 is composed of two different shapes, a square and an ellipse, included in Ω . We choose the simple case of $f = 1$. The exact solution is computed with $\beta_1 = 5.0$ and $\beta_2 = 1.0$ and we choose $r = 1.0$ and $w = 5 \cdot 10^{-6}$ and obtain a very accurate recovery of the coefficients and the interface, see Table 1 and Figure 2.

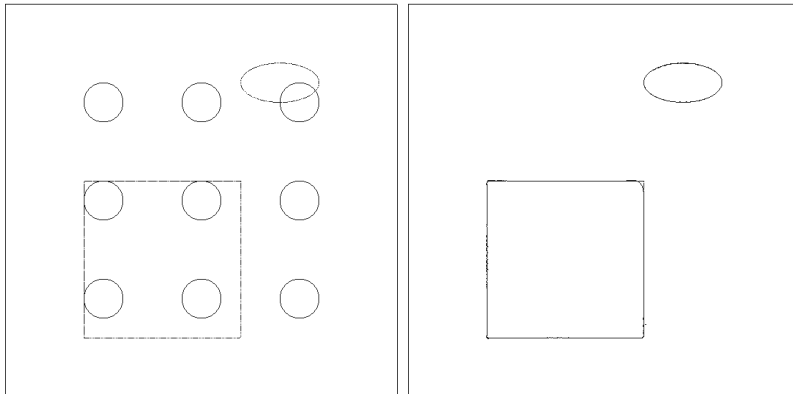


FIG. 2. *Interface recovery for Poisson's equation: exact solution with initial (left) and final configuration (right).*

coefficient	exact	recovered	relative error
β_1	5.0	5.00252	0.0504 %
β_2	1.0	1.00089	0.089 %

TABLE 1

Recovered coefficients for Poisson's equation.

4.2. Linear Elasticity. We focus our numerical experiments on examples in linear elasticity and present multiple settings: we use the same geometric setting as

before, with and without the addition of uniformly distributed noise, and also employ a more elaborate geometry. For all these examples, f is set equal to zero.

4.2.1. Square and ellipse. For this example, we choose the same geometric setting as above, with Ω_1 being a square and an ellipse. The boundary conditions are given as pure stretching by $u = (-0.01, 0)$ on the left and $u = (0.01, 0)$ on the right, and free on top and bottom (see Figure 3 for an illustration). The material parameters are set to $E_1 = 200.0$, $\nu_1 = 0.28$ and $E_2 = 117.0$, $\nu_2 = 0.33$, for which we choose the initial guesses $E_1 = E_2 = 150.0$, $\nu_1 = \nu_2 = 0.30$. We choose $r = 0.001$, $w = 10^{-6}$ and are able to recover the coefficients (see Table 2) and the interface (see Figure 4 for the evolution) very accurately for this case.

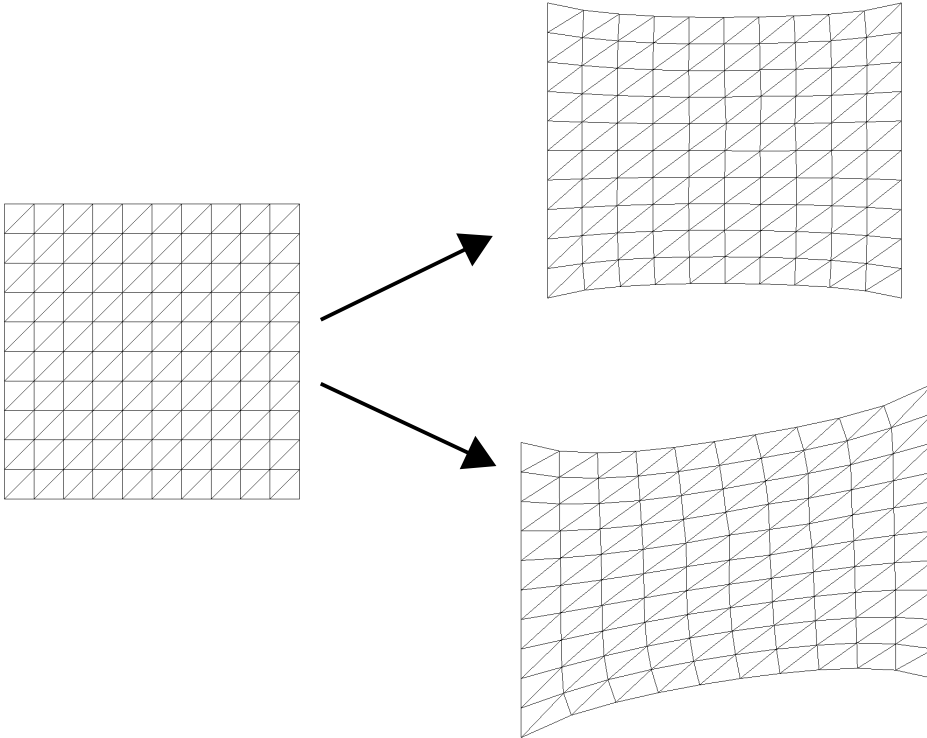


FIG. 3. Illustration of the boundary displacements used in the examples: the undeformed domain on the left is deformed into the configuration on the right, either by pure stretching in the example square and ellipse (upper right) or by pulling and shearing in the example elaborate geometry (lower right). Note that the boundary displacements used in this picture are exaggerated for illustration purposes, the displacements used in the experiments are much smaller.

4.2.2. Square and ellipse with noise. We now use the same setting as described above and add noise to the exact solution. We obtain our observation u_0 as follows:

$$u_0 = u^* + \sigma \frac{\|u^*\|_{L^2}}{\|S\|_{L^2}} S,$$

where u^* is the exact solution for a given set of coefficients. The vector S contains nodal values $s_i \in \mathbb{R}^d$, with $(s_i)_j \in [-1, 1]$ uniformly random, and σ controls the noise level. See Table 3 and Figure 5 for the results with $\sigma = 5\%$, $w = 10^{-5}$ and $r = 0.005$.

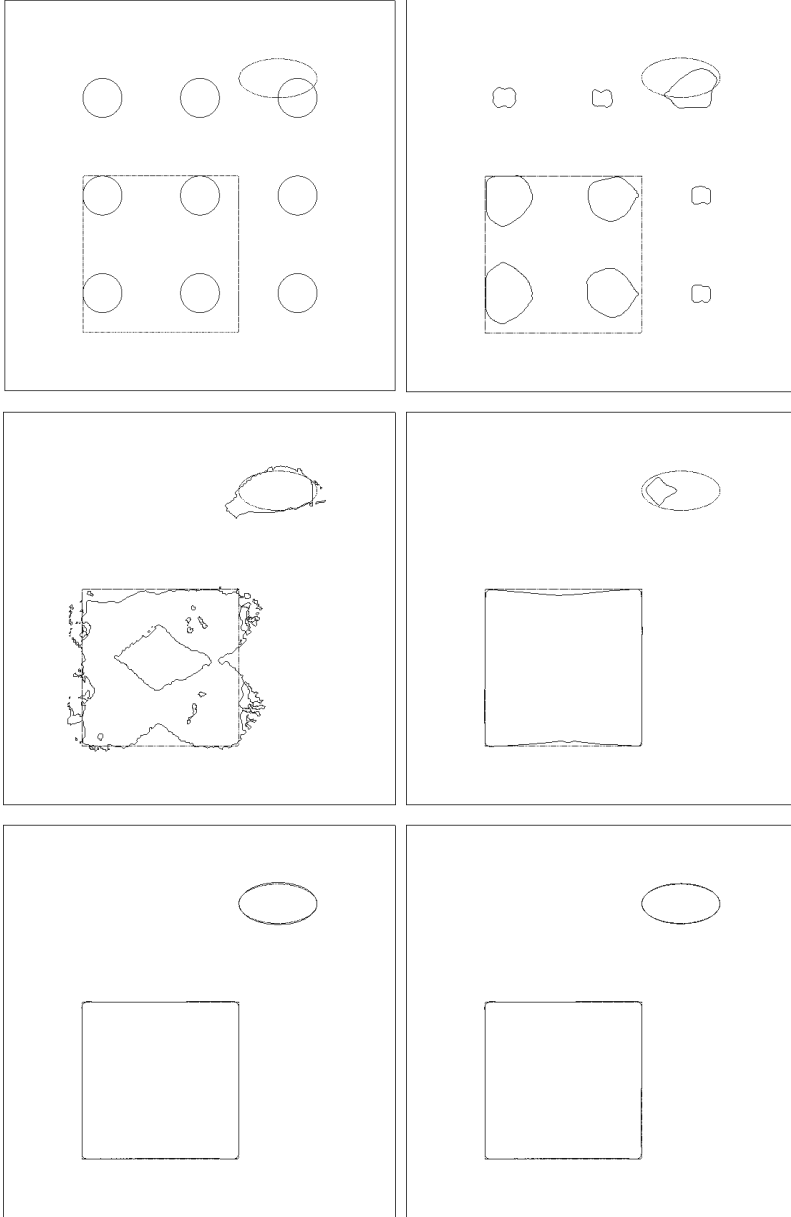


FIG. 4. Interface evolution for linear elasticity example square and ellipse: after 0, 1000, 2000, 5000, 10000, 12000 iterations (from top left to bottom right).

4.2.3. Elaborate geometry. In our last example, we show that we can target a more complicated geometry as shown in Figure 6 on the *left*. The material properties are given as in the previous example, but we change the initial guesses to $E_1 = 230.0$, $\nu_1 = 0.30$ and $E_2 = 80.0$, $\nu_2 = 0.35$, the boundary conditions are specified as $u = (-0.02, -0.01)$ on the left and $u = (0.02, 0.01)$ on the right, and free on top and bottom, i.e. we pull and shear the material (see Figure 3 for an illustration). For this example, we set $w = 10^{-5}$ and $r = 0.004$. The results can be seen in Table 4 and

coefficient	exact	recovered	relative error
E_1	200.0	200.068	0.034 %
ν_1	0.28	0.280609	0.2175 %
E_2	117.0	117.547	0.4675 %
ν_2	0.33	0.329976	0.0073 %

TABLE 2

Recovered coefficients for linear elasticity example square and ellipse.

coefficient	exact	recovered	relative error
E_1	200.0	190.139	4.9305 %
ν_1	0.28	0.280381	0.1361 %
E_2	117.0	111.176	4.9778 %
ν_2	0.33	0.330018	0.0055 %

TABLE 3

Recovered coefficients for linear elasticity example square and ellipse, with $\sigma = 5\%$ noise.

Figure 6 and show how accurately we can recover coefficients and interface even in this setting.

coefficient	exact	recovered	relative error
E_1	200.0	207.974	3.9870 %
ν_1	0.28	0.281306	0.4664 %
E_2	117.0	121.568	3.9042 %
ν_2	0.33	0.330681	0.2063 %

TABLE 4

Recovered coefficients for linear elasticity example elaborate geometry.

5. Conclusion. We have presented a fast, accurate and easy to implement algorithm to solve for the arising coefficients as well as the discontinuity interface in elliptic PDEs, demonstrating it for the problems of Poisson’s equation and linear elasticity. The advantage of our approach is the treatment of the coefficients as truly piecewise constant. This provides us with a fast explicit update at every iteration since we only need one linear solve at every iteration, all other variables are updated explicitly and we do not need an additional non-linear solve. Further, our approach is general and can be employed to any linear elliptic inverse problem. As presented in this paper, this allows us to provide the simultaneous recovery of coefficients and interface for Poisson’s equation and linear elasticity. For the case of Poisson’s equation we have to assume a non-zero right hand side but for linear elasticity the forcing term can be of any nature. Several numerical examples show that our method can handle a variety of geometries, and the addition of noise.

We expect our method to work with experimental data since we demonstrated the ability to handle input data polluted with noise. In the case of medical imaging, the interior measurement u_0 could be acquired by techniques involving ultrasound or MRI, see [35, 38, 45, 46, 56] and references therein.

Currently, our method is based on a first order accurate solver on a triangular

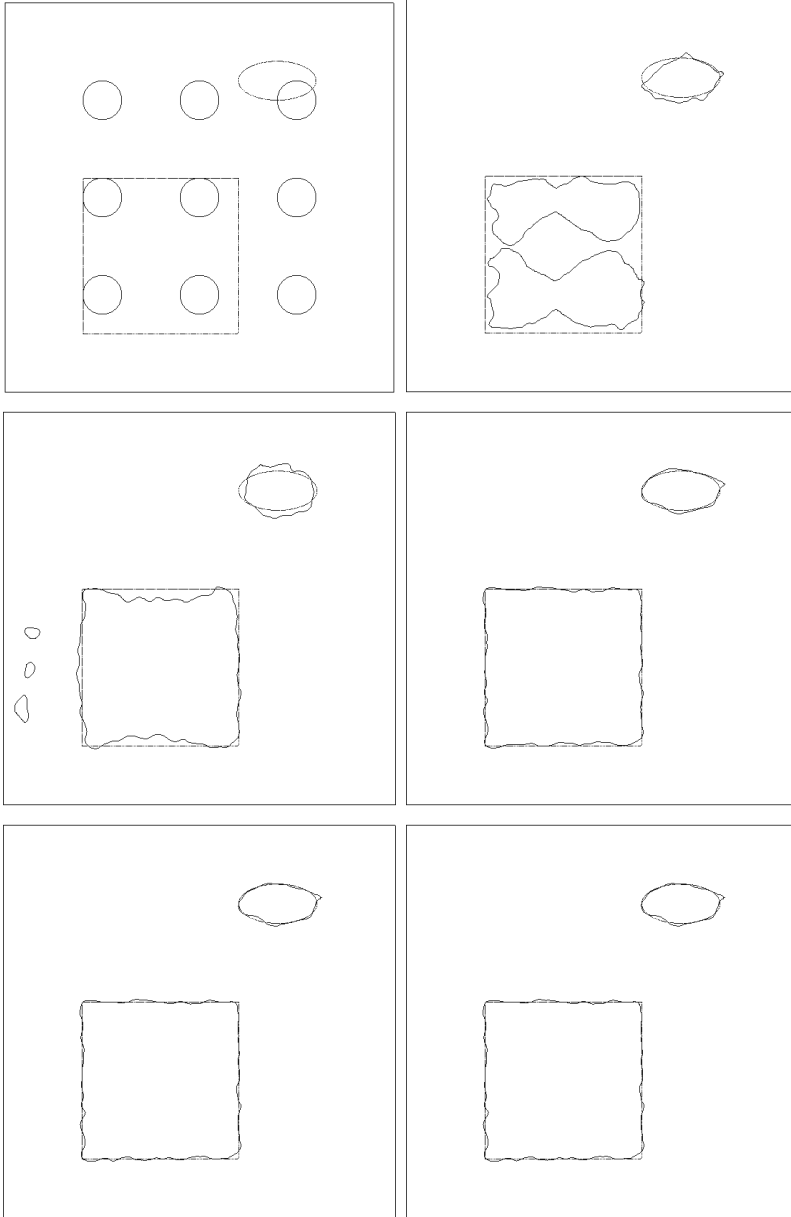


FIG. 5. Interface evolution for linear elasticity example square and ellipse, with $\sigma = 5\%$ noise: after 0, 1000, 3000, 5000, 10000, 13000 iterations (from top left to bottom right).

mesh. The incorporation of a second order method (cf. [6, 42, 67]) could potentially enhance every iteration. Also, a more advanced preconditioner, possibly based on multigrid methods (see [44] and the references therein), could further improve the runtime performance.

Appendix A. Shape Derivatives.

For completeness, we state some brief information about shape derivatives, which

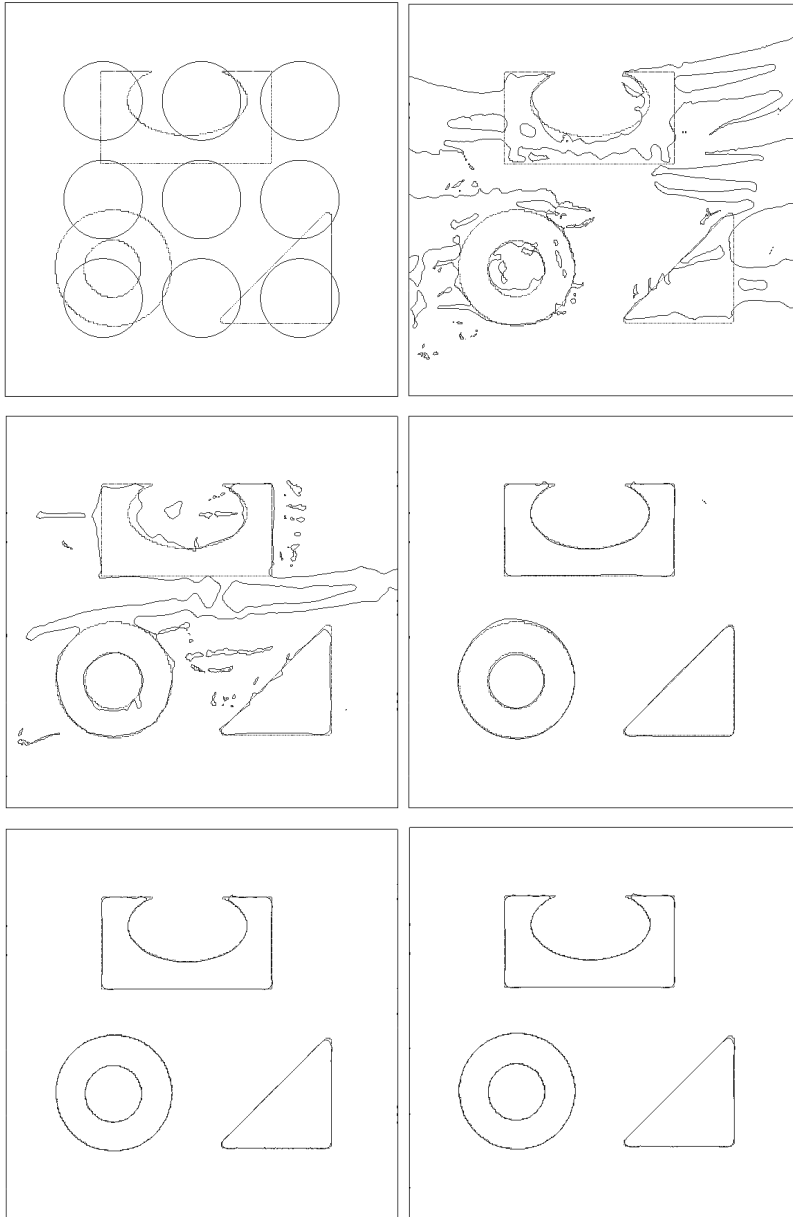


FIG. 6. *Interface evolution for linear elasticity example elaborate geometry: after 0, 1000, 2000, 3000, 4000, 5000 iterations (from top left to bottom right).*

we use in Section 2 to compute the derivatives of the energy functional with respect to the geometry. Following [2], we apply the subsequent statements (cf. [61]) in order to derive the correct speed for the interface evolution.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth open set and and $\varphi \in W^{1,1}(\mathbb{R}^d)$. Then*

$$J(\Omega) := \int_{\Omega} \varphi(x) dx$$

is differentiable at Ω and

$$\begin{aligned} J'(\Omega)[\theta] &= \int_{\Omega} \nabla \cdot (\theta(x)\varphi(x)) dx \\ &= \int_{\partial\Omega} \theta(x) \cdot n(x)\varphi(x) ds \end{aligned}$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$.

LEMMA 2. Let $\Omega \subset \mathbb{R}^d$ be a smooth open set and $\varphi \in W^{2,1}(\mathbb{R}^d)$. Then

$$J(\Omega) := \int_{\partial\Omega} \varphi(x) ds$$

is differentiable at Ω and

$$J'(\Omega)[\theta] = \int_{\partial\Omega} \theta(x) \cdot n(x) \left(\frac{\partial\varphi}{\partial n}(x) + \varphi(x)\kappa(x) \right) ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$, where $\kappa = \nabla \cdot n$ is the mean curvature of $\partial\Omega$.

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