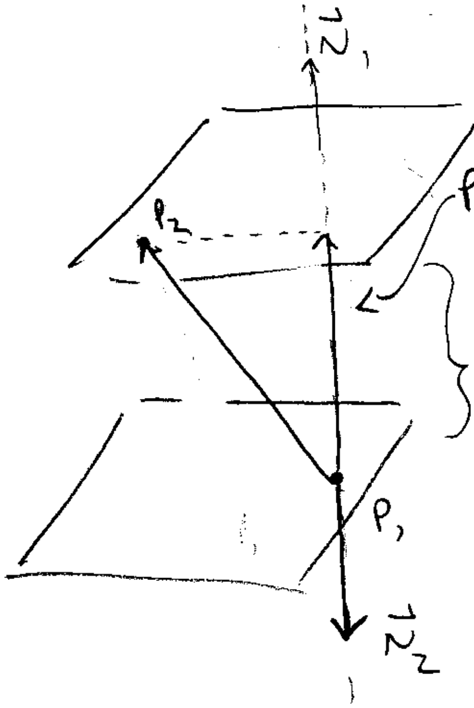
Practice Final 1 : Solutions

1 The planes are parallel so all we need to do is pick two points on each, and project the vector between them onto either normal of the given planes.

$5x+y-z=1$

$10x+2y-2z=5$



b/c $\vec{N}_1$ is parallel to $\vec{N}_2$

$\text{proj}_{N_2}\overrightarrow{P_1P_2} = \text{proj}_{N_1}\overrightarrow{P_1P_2}$

distance $= \text{comp}_{N_1}\overrightarrow{P_1P_2}$

So, to find points just set $x, y = 0$ in each and solve for z.

$\Rightarrow -2z = 5 \Rightarrow z = -5/2$ so $P_1 = (0, 0, -5/2)$

$\Rightarrow -z = 1 \Rightarrow z = -1$ so $P_2 = (0, 0, -1)$

Then, the vector between them is $\langle 0, 0, -3/2 \rangle$

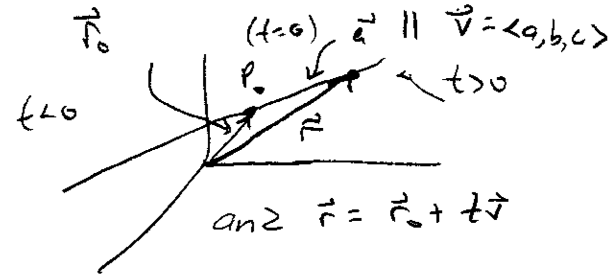
Project this onto $\vec{N}_1$ so

$$\text{comp}_{\vec{N}_2}\overrightarrow{P_1P_2} = \frac{|\vec{N}_2 \cdot \overrightarrow{P_1P_2}|}{|\vec{N}_2|} = \frac{|\langle 5, 1, -1\rangle \cdot \langle 0, 0, -3/2\rangle|}{\sqrt{5^2+1^2+1^2}}$$

$$= \frac{3/2}{\sqrt{27}} = \frac{3/2}{3\sqrt{3}} = \frac{\sqrt{3}}{6} \quad \text{II}$$

1) a) $\langle 8,4,1\rangle$ is a vector representing the direction of the line and when we parametrize w.r.t. $(3,4,5)$ we get

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$



Where $\langle a,b,c\rangle$ is the vector parallel to the lines directi and (x_0, y_0, z_0) is a point on the line.

So
$$x(t) = 3 + 8t$$
$$y(t) = 4 + 4t \quad \Rightarrow \quad \vec{r}(t) = \langle 3+8t, 4+4t, 5+t\rangle$$
$$z(t) = 5 + t$$

(b) Recall the arclength function measured from some point corresponding to a time value of a is

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

So $$\frac{ds}{dt} = |r'(t)|$$

Our point is $(3,4,5)$ so we need to know what t value this corresponds to. $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$
$= \langle 3+8t, 4+4t, 5+t\rangle$

So clearly $t=0$ works.

Now, $\vec{r}\,'(t) = \langle 8, 4, 5\rangle$ so $|\vec{r}\,'(t)| = \sqrt{64+16+25} = \sqrt{105}$

Thus, $\frac{ds}{dt} = \sqrt{105}$ and $s = s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{105}\,du$

$= \sqrt{105}\,t$

So $t = t(s) = s/\sqrt{105}$

Thus, $\vec{r}(t(s))$ is a function of arclength s, not t

and $\vec{r}(t(s)) = \left\langle 3 + \frac{8s}{\sqrt{105}},\ 4 + \frac{4s}{\sqrt{105}},\ 5 + \frac{s}{\sqrt{105}} \right\rangle$ $\square$

3) $\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t, 3t^2, 4t^3\rangle + \vec{c}$

constants from each coordinate integral

$\vec{v}(0) = \langle 1,0,0\rangle = \langle 0,0,0\rangle + \langle c_1, c_2, c_3\rangle$

$\Rightarrow c_1 = 1, \quad c_2 = c_3 = 0$

$\Rightarrow \vec{v}(t) = \langle 2t+1, 3t^2, 4t^3\rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2 + t, t^3, t^4\rangle + \vec{k}$

$\vec{r}(0) = \langle 0, 1, -1\rangle = \langle 0,0,0\rangle + \langle k_1, k_2, k_3\rangle$

$\Rightarrow k_1 = 0, \quad k_2 = 1, \quad k_3 = -1.$

So, $\vec{r}(t) = \langle t^2 + t, t^3 + 1, t^4 - 1\rangle$ $\square$

$$) \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r}\right] = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

$$= \cos\theta\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right]$$

$$+ \sin\theta\left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right]$$

$$= \cos\theta\left[\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y \partial x}\sin\theta\right]$$

$$+ \sin\theta\left[\frac{\partial^2 z}{\partial x \partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right]$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta + \underbrace{\frac{\partial^2 z}{\partial x \partial y} 2\sin\theta\cos\theta}_{\frac{\partial^2 z}{\partial x \partial y}\sin(2\theta)?}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$$

$$\frac{\partial^2 z}{\partial \theta^2} = -r\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\sin\theta\right) + r\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\cos\theta\right)$$

$$= -r\left[\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial \theta}\right)\sin\theta + \frac{\partial z}{\partial x}\cos\theta\right]$$

$$+ r\left[\left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta}\right)\cos\theta - \frac{\partial z}{\partial y}\sin\theta\right]$$

$$= -r\left[\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y \partial x}(r\cos\theta)\right)\sin\theta + \frac{\partial z}{\partial x}\cos\theta\right]$$

$$+ r\left[\left(\frac{\partial^2 z}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2}(r\cos\theta)\right)\cos\theta - \frac{\partial z}{\partial y}\sin\theta\right]$$

$$= r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - r^2\sin\theta\cos\theta\,\frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial x}r\cos\theta$$

$$+ -r^2\sin\theta\cos\theta\,\frac{\partial^2 z}{\partial x \partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y}r\sin\theta$$

So, $$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = f_{xx}\cos^2\theta + f_{yy}\sin^2\theta + 2f_{xy}\sin\theta\cos\theta$$

$$+ f_{xx}\sin^2\theta - f_{yx}\sin\theta\cos\theta - \frac{f_x\cos\theta}{r}$$

$$- f_{xy}\sin\theta\cos\theta + f_{yy}\cos^2 - \frac{f_y\sin\theta}{r}$$

$$+ \frac{f_x\cos\theta}{r} + \frac{f_y\sin\theta}{r}$$

$$= f_{xx} + f_{yy} \quad \square$$

5) The maximal directional derivative is $|\nabla f|$ since

$$D_u f(x) = \nabla f\cdot\vec{u} \quad \text{and}$$

$$|\nabla f\cdot\vec{u}| \leq |\nabla f||u| = |\nabla f|$$

or $\nabla f\cdot\vec{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta \leq |\nabla f|$ max when $\theta = 0$ i.e. $\vec{u} \parallel \nabla f$

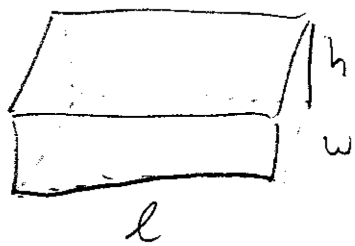
So $$\nabla f = \left\langle \frac{-y^2}{x^2}, \frac{2y}{x}\right\rangle$$

So $$\nabla f|_{(2,4)} = \left\langle \frac{-4^2}{2^2}, \frac{2\cdot 4}{2}\right\rangle = \langle -4, 4\rangle$$

$$|\nabla f|_{(2,4)}| = \sqrt{2\cdot 16} = 4\sqrt{2}.$$

It occurs in the direction of $\nabla f = \langle -4, 4 \rangle$
which as a unit vector is $= \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$. $\square$

6)



$$lhw = 32000$$
$$\Rightarrow h = \frac{32000}{lw}$$

$$SA = 2lh + 2lw + 2hw$$
$$= 2(lh + lw + hw)$$
$$= 2\left(\frac{32000}{w} + lw + \frac{32000}{l}\right)$$

So $\nabla SA = 2\langle (w - 32000/l^2), (-32000/w^2 + l) \rangle$

$$= \langle 0, 0 \rangle$$

$\Rightarrow w = 32000/l^2$ and $l = 32000/w^2$

$\Rightarrow l^2 w = 32000 = lhw$ $\qquad$ $\Rightarrow lw^2 = 32000 = lhw$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Rightarrow h = w$

$\Rightarrow h = l = w$

So $w^3 = 32000 \Rightarrow w \approx 31.748\ldots$ $\square$

7) $\nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

So
$$xy^2z^2 = \lambda x$$
$$yx^2z^2 = \lambda y$$
$$zx^2y^2 = \lambda z$$

If $x \neq 0$, $y \neq 0$, $z \neq 0$ then $y^2z^2 = \lambda \Rightarrow yx^2z^2 = y^3z^2$
$\Rightarrow x^2 = y^2$
$\Rightarrow x = \pm y$

Then $2y^2y^2 = \lambda z = y^2z^2z$

$\Rightarrow 2y^4 = y^2z^3$

$\Rightarrow y^2 = z^2 \Rightarrow z = \pm y = \pm x$

$x^2+y^2+z^2 = 1$

$\Rightarrow 3y^2 = 1 \Rightarrow y = \pm 1/\sqrt{3}$

$f(x,y,z) = x^2y^2z^2$ so sign doesn't matter, $f \geq 0$ always, so clearly 0 is min; set $x=0$ and y, z anything s.t. $y^2+z^2=1$ (or with y & z)

Max is then at $x=y=z = \pm 1/\sqrt{3}$ giving $(1/3)^3 = 1/27$. □

8) (a) ∇f is always $\perp$ to the level sets of f b/c it is the direction f is increasing the fastest. Thus, if u is tangent to S then $D_u f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u} = 0$

(b) Then $\nabla f \cdot \vec{u} = |\nabla f|\,|u| \cos\theta = |\nabla f|$ b/c $|u|=1$ and $\theta = 0$ or π b/c both are $\perp$ to S, or rather to the tangent plane of S.

So $\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{4+9+16}$
$= \sqrt{29}$ □

1) f has a local min if $f_x(0,0) = 0 = f_y(0,0)$

and $f_{xx}(00)\, f_{yy}(0,0) - [f_{xy}(0,0)]^2 > 0$

with $f_{xx}(0,0) > 0$ (p, 960 of text)

2^{nd} Deriv test

Well, $f_x = 2x + ky$

$f_{xx} = 2$ $\quad f_{xy} = k$

$f_y = kx + 6y$

$f_{yy} = 6$

So, $f_x(0,0) = 0$ $\quad f_{xy}(0,0) = k$

$f_y(0,0) = 0$

$f_{xx}(0,0) = 2$

$f_{yy}(0,0) = 6$

So, $f_{xx} f_{yy} - (f_{xy})^2 = 12 - k^2 > 0$

and $f_{xx} > 0$ already $\Rightarrow 12 > k^2$

$\Rightarrow k < \sqrt{12}$

$= 2\sqrt{3}.$ □

finally, to maximize

$f(1,0) = 2$ is large

b/c any amount you give to y is made smaller by the 4th power Then the 3rd power of x makes it and also you multiply by 2 with x.

0) First find critical values of f on D.

$f_x = 6x^2 = 0 \iff x = 0$ $\quad$ f = 0 at these values

$f_y = 4y^3 = 0 \iff y = 0$

on boundary though, $f(-1,0) = -2$, smallest

b/c gives most negative x component, and any positive or neg y just makes f less negative