Lines + Planes

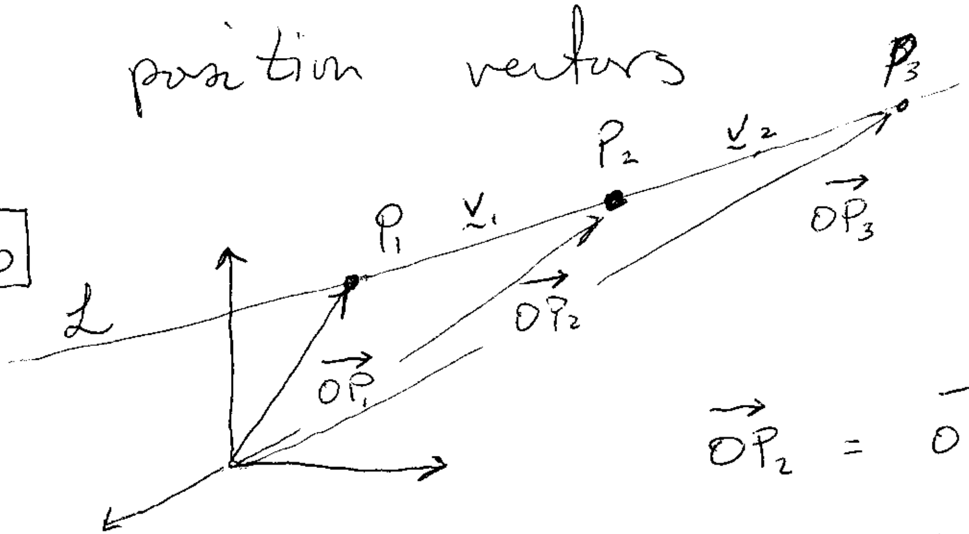
10/6/08

①

1a - lines, planes are collections of points in space

- here, we associate points w/ position vectors

1b



$$\vec{OP_2} = \vec{OP_1} + \underset{\sim}{v}_1$$

$$\vec{OP_3} = \vec{OP_1} + \underset{\sim}{v}_1 + \underset{\sim}{v}_2$$
$$= \vec{OP_2} + \underset{\sim}{v}_2$$

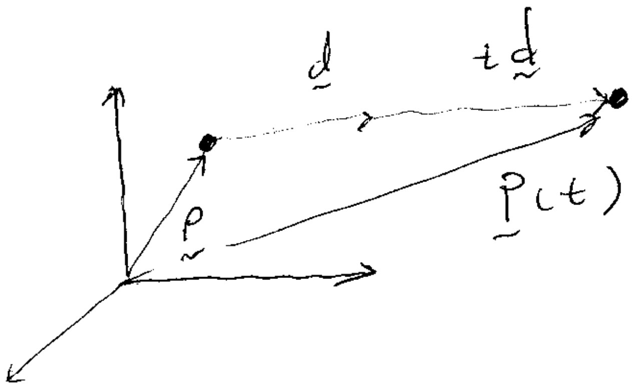
1b $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are parallel vectors

(so is $\underset{\sim}{v}_1 + \underset{\sim}{v}_2$)

- $$\frac{\underset{\sim}{v}_1}{|\underset{\sim}{v}_1|} = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|} = \frac{\underset{\sim}{v}_1 + \underset{\sim}{v}_2}{|\underset{\sim}{v}_1 + \underset{\sim}{v}_2|}$$

Parametric Equation for the line $\mathcal{L}$ through $\underset{\sim}{p}$ w/ direction $\underset{\sim}{d}$

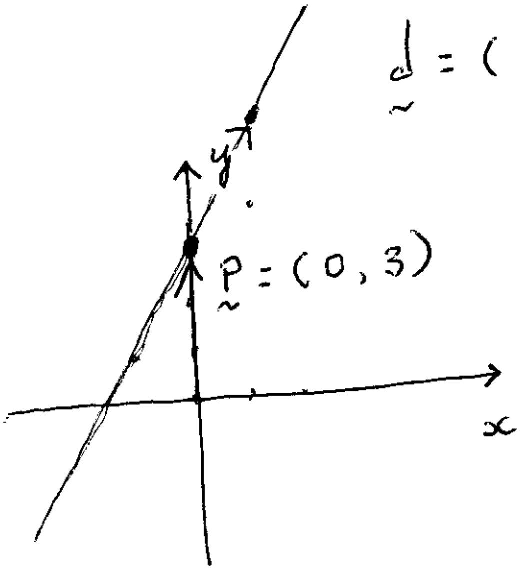
$$\underset{\sim}{p}(t) = \underset{\sim}{p} + t\,\underset{\sim}{d}$$ – t parameterizes the line



$\underset{\sim}{d} = (1, 2)$

$\boxed{1b}$ Example

$y = 2x + 3$



$$\underset{\sim}{p}(t) = (0, 3) + t\,(1, 2)$$

$$= (t, 3 + 2t)$$

$x(t) = t$

$y(t) = 3 + 2t$

i.e. $y(x) = 3 + 2x$

b] **Direction Numbers**

$$\underset{\sim}{p}(t) = \underset{\sim}{p} + t\,\underset{\sim}{d}$$

$$= \left(p_1 + td_1,\ p_2 + td_2,\ p_3 + td_3\right)$$

d_1, d_2, d_3 = direction numbers

b]
* non-unique *

$\underset{\sim}{p}(t) = \underset{\sim}{p} + t\,\underset{\sim}{d}$ describe the same line

$+\ \underset{\sim}{p}(t) = \underset{\sim}{p} + \cancel{t}\,(\lambda \underline{d})$

so (d_1, d_2, d_3) and $(\lambda d_1, \lambda d_2, \lambda d_3)$ are both valid direction numbers for the **same** line

Symmetric Equations

$$\underset{\sim}{p}(t) = \underset{\sim}{p} + t\,\underset{\sim}{d}$$

$$= (p_1 + td_1,\ p_2 + td_2,\ p_3 + td_3)$$

i.e.

$$x = p_1 + t d_1 \quad \longleftrightarrow \quad \frac{x - p_1}{d_1} = t$$

$$y = p_2 + t d_2 \quad \longleftrightarrow \quad \frac{y - p_2}{d_2} = t$$

$$z = p_3 + t d_3 \quad \longleftrightarrow \quad \frac{z - p_3}{d_3} = t$$

$$\therefore \quad \frac{x - p_1}{d_1} = \frac{y - p_2}{d_2} = \frac{z - p_3}{d_3}$$

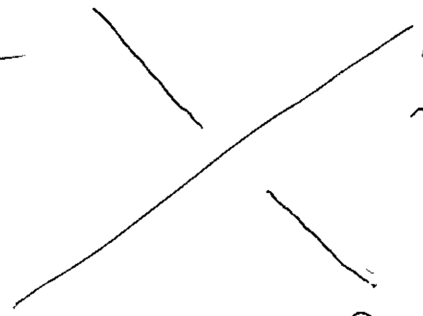
symmetric equations for the line

* if, e.g. $d_1 = 0$, then

$$x = p_1, \quad \frac{y - p_2}{d_2} = \frac{z - p_3}{d_3}$$

rb] **Skew Lines**

- non-parallel, non-intersecting lines



$\underset{\sim}{P_1}(t) = \underset{\sim}{P_1} + t\,\underset{\sim}{d_1}$

$\underset{\sim}{P_2}(t) = \underset{\sim}{P_2} + t\,\underset{\sim}{d_2}$

so $\dfrac{\underset{\sim}{d_1}}{|\underset{\sim}{d_1}|} \neq \dfrac{\underset{\sim}{d_2}}{|\underset{\sim}{d_2}|}$ and there is

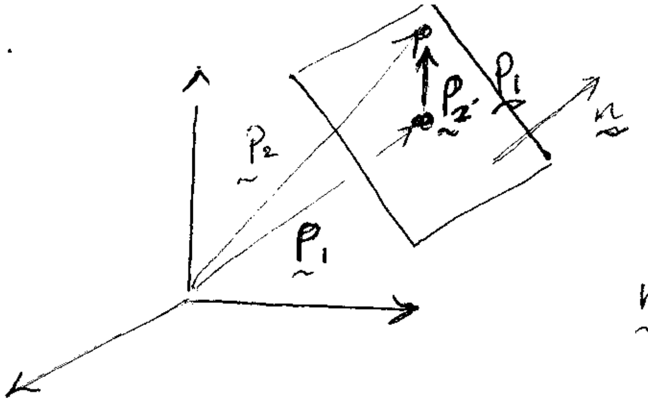
no (s,t) s.t. $\underset{\sim}{P_1}(t) = \underset{\sim}{P_2}(S)$

b] **Planes**

- can be determined by one point ($\underline{P}$) in the plane and a vector $\underset{\sim}{n}$ <u>normal</u> to the plane

- $\underset{\sim}{n}$ is orthogonal to the plane

1b



$$\underset{\sim}{n} \cdot (\underset{\sim}{p_2} - \underset{\sim}{p_1}) = 0$$

- so if $\underset{\sim}{p_1}$ and $\underset{\sim}{n}$ are a point in the plane and the normal to the plane, then $\underset{\sim}{p}$ is also in the plane if

$$\underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{p_1}) = 0$$

1b i.e. if $\underset{\sim}{n} = (n_1, n_2, n_3)$, then a point $\underset{\sim}{p} = (x, y, z)$ is in the plane if

$$n_1 (x - x_1) + n_2 (y - y_1) + n_3 (z - z_3) = 0$$

or $n_1 x + n_2 y + n_3 z = n_1 x_1 + n_2 x_2 + n_3 x_3$

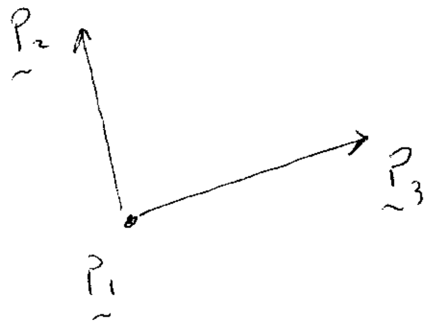
e.g.

$2x + 5y + 3z = 4$ is the equation for a plane

Example

\boxed{15} Three non-colinear points determine a plane.

$\underset{\sim}{P}_1, \underset{\sim}{P}_2, \underset{\sim}{P}_3$



$$\underset{\sim}{n} = (\underset{\sim}{P}_3 - \underset{\sim}{P}_1) \times (\underset{\sim}{P}_2 - \underset{\sim}{P}_1)$$

e.g. $\underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{p}_1) = 0$

is an equation for the plane

\boxed{16} ## Parallel Planes

- two planes intersect in a line
- or are parallel (i.e. $\underset{\sim}{n}_1 = \underset{\sim}{n}_2$)

* skew lines lay in parallel planes

$\mathcal{L}_1 : \underset{\sim}{\ell}_1(t) = \underset{\sim}{p}_1 + t\,\underset{\sim}{d}_1$

$\underset{\sim}{n} = \underset{\sim}{d}_1 \times \underset{\sim}{d}_2$

$\mathcal{L}_2 : \underset{\sim}{\ell}_2(t) = \underset{\sim}{p}_2 + t\,\underset{\sim}{d}_2$

$P_1 : \underset{\sim}{n} \cdot (\overset{\underset{\sim}{p}}{\text{[illegible]}} - \underset{\sim}{p}_1)$

$P_2 : \underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{p}_2)$

rb· check

$$P_1 : \quad \underset{\sim}{n} \cdot (\underset{\sim}{l}_1(t) - \underset{\sim}{p}_1) = \underset{\sim}{n} \cdot (\underset{\sim}{p}_1 + t\underset{\sim}{d}_1 - \underset{\sim}{p}_1)$$

$$= \underset{\sim}{n} \cdot (t\underset{\sim}{d}_1) = 0$$

$$P_2 : \quad \underset{\sim}{n} \cdot (\underset{\sim}{l}_2(t) - \underset{\sim}{p}_2) = \underset{\sim}{n} \cdot (\underset{\sim}{p}_2 + t\underset{\sim}{d}_2 - \underset{\sim}{p}_2)$$

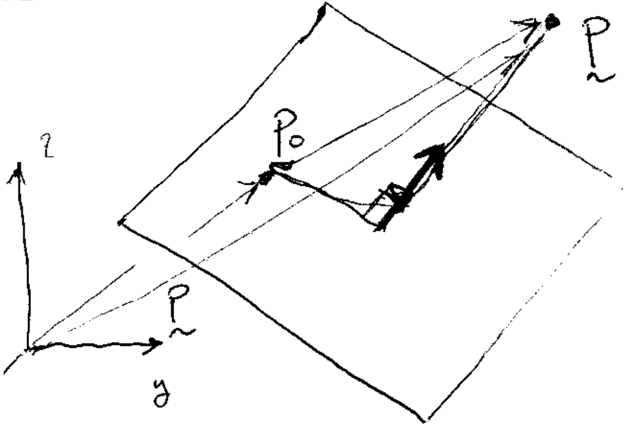
$$= \underset{\sim}{n} \cdot t\underset{\sim}{d}_2 = 0$$

Mb **Summary**

- P_1 and P_2 are parallel planes
- $\underset{\sim}{l}_1$ lies in P_1 and
  $\underset{\sim}{l}_2$ lies in P_2
- the normal to P_1 + P_2 is orthogonal to both $\underset{\sim}{d}_1$ and $\underset{\sim}{d}_2$

0b

Distance to a plane



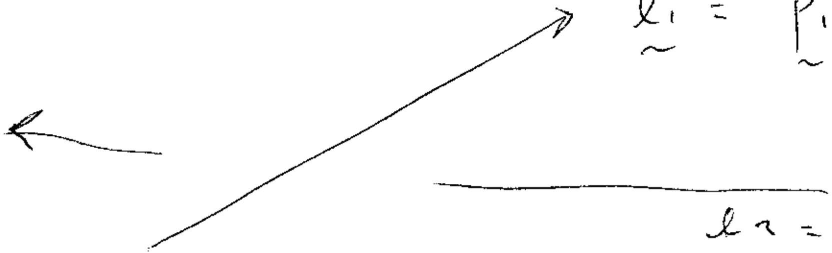
$P : \underset{\sim}{n} , \underset{\sim}{p}_0$

$$\text{distance} = \left| \text{comp}_{\underset{\sim}{n}} \overset{\underset{\sim}{p}-\underset{\sim}{p}_0}{\underset{\sim}{\text{[illegible]}}} \right|$$

$$= \left| \frac{\underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{p}_0)}{|\underset{\sim}{n}|} \right|$$

11b

Distance between skew lines



$\underset{\sim}{l}_1 = \underset{\sim}{p}_1 + t \underset{\sim}{d}_1$

$\underset{\sim}{l}_2 = \underset{\sim}{p}_2 + t \underset{\sim}{d}_2$

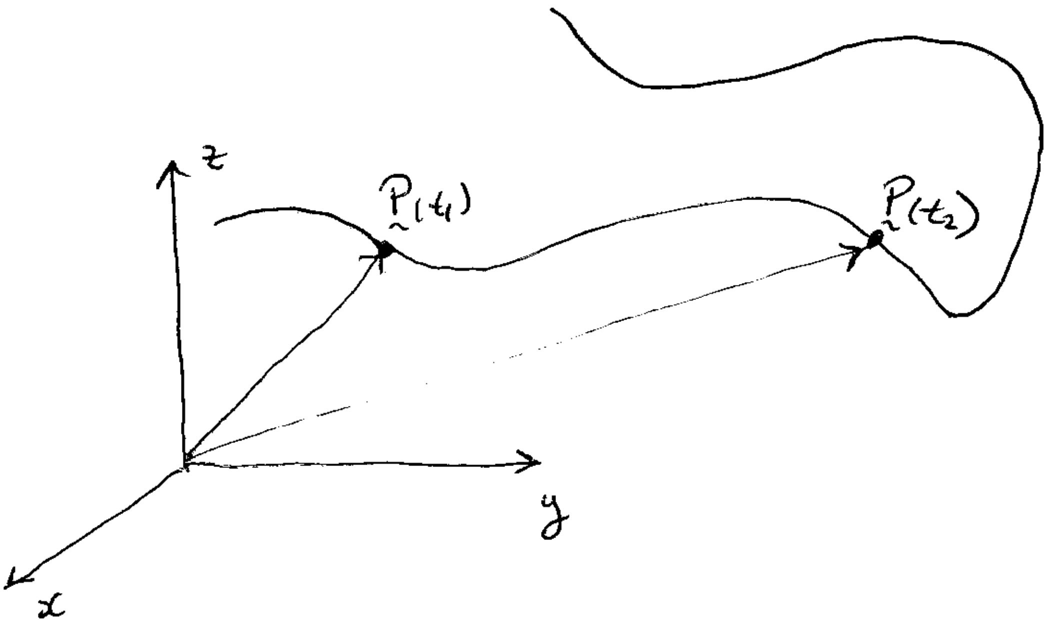
- compute parallel plane

  e.g. $P_1 : (\underset{\sim}{d}_1 \times \underset{\sim}{d}_2) \cdot (\underset{\sim}{p} - \underset{\sim}{p}_1)$

- compute distance from point $\underset{\sim}{p}_2$ to this plane

11.1 Parametric Equations

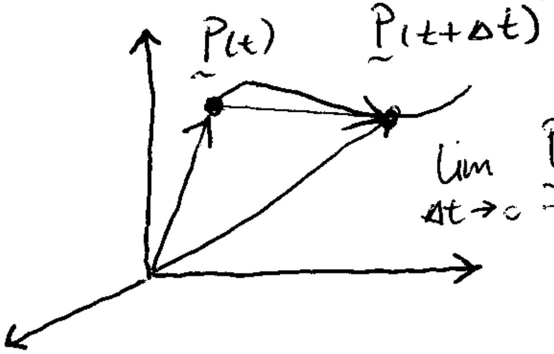
- curves in space and the plane
- position vector as a function of a parameter: $\underset{\sim}{p}(t) = (x(t), y(t), z(t))$



e.g. trajectory of a particle through space

- derivative of this function is the particle <u>velocity</u>

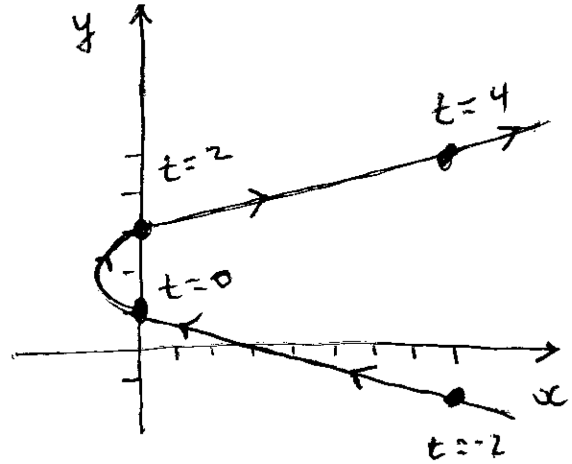
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



$$\lim_{\Delta t \to 0} \frac{\underset{\sim}{P}(t+\Delta t) - \underset{\sim}{P}(t}{\Delta t}$$

E.g.

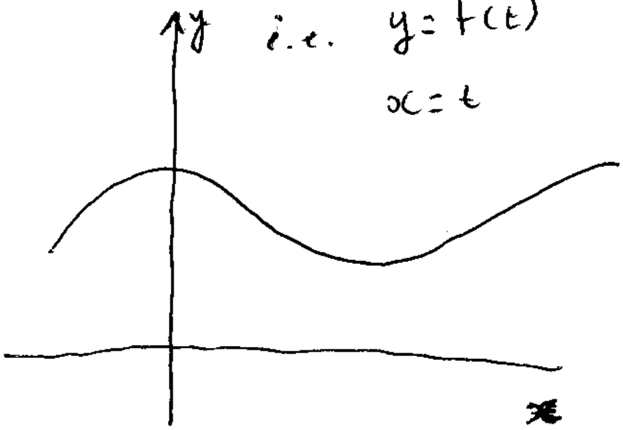
Ia $x(t) = t^2 - 2t$, $y(t) = t+1$



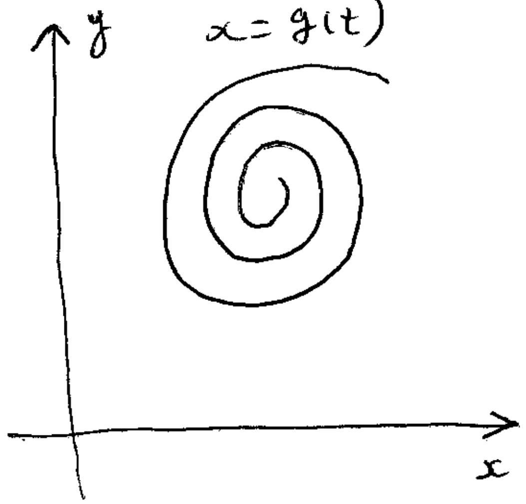
t	$x(t)$	$y(t)$
-2	8	-1
0	0	1
2	0	3
4	8	5

Ib $y = f(x)$

i.e. $y = f(t)$
$x = t$

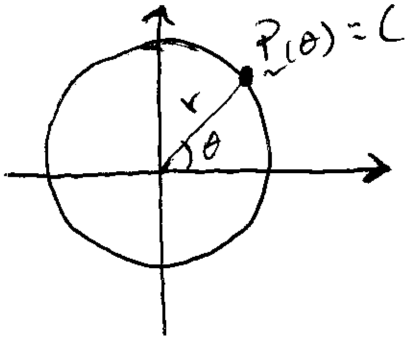


$y = f(t)$
$x = g(t)$



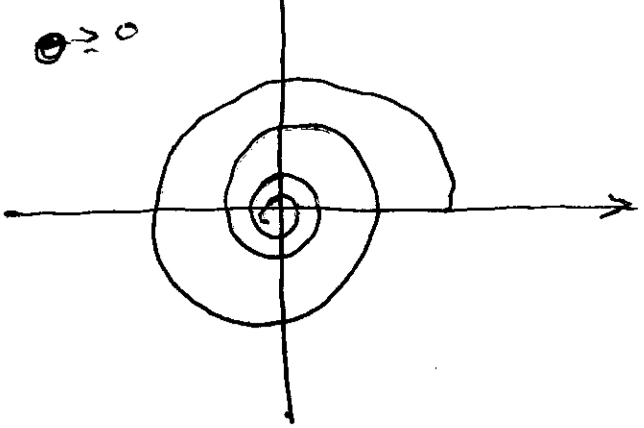
IIb Polar coordinates

$\underset{\sim}{P}(\theta) = (r\cos(\theta), r\sin(\theta))$



$\underset{\sim}{P}(\theta) = (e^{-\theta}\cos(\theta), e^{-\theta}\sin(\theta))$

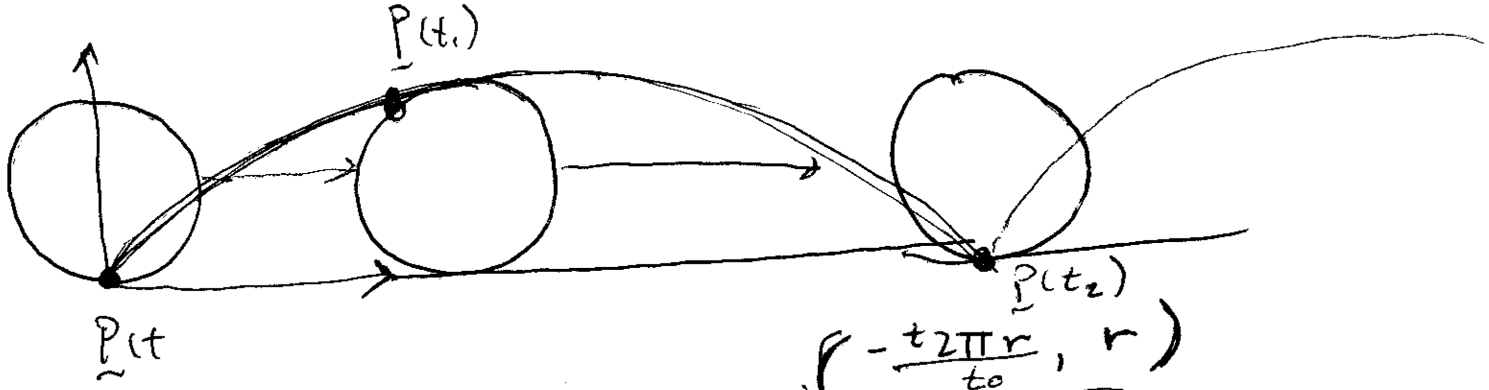
$\theta \geq 0$



Ib

E.g.

Cycloid

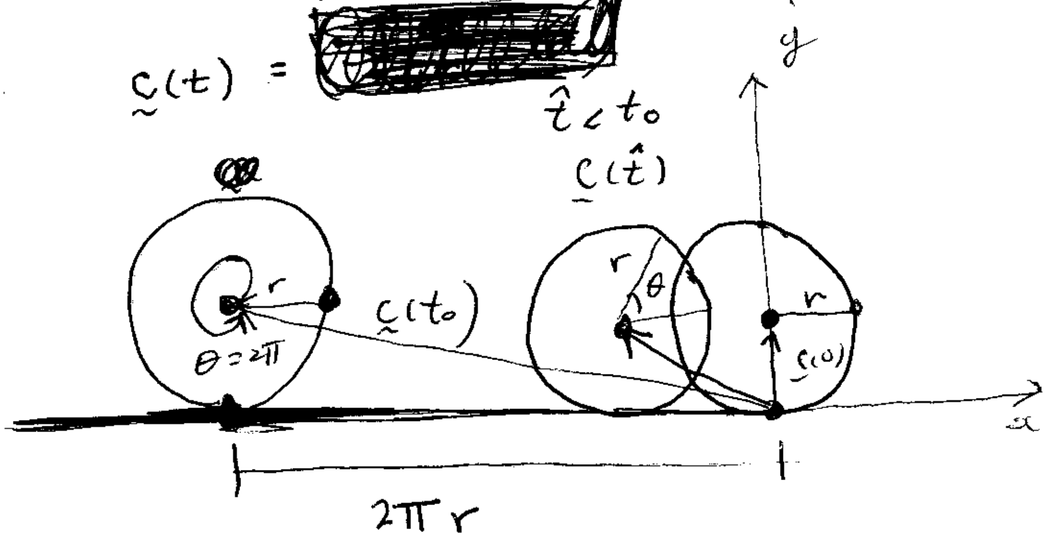


$$\underset{\sim}{c}(t) = \left(-\frac{t\,2\pi r}{t_0},\ r \right)$$

[redacted]

IIb

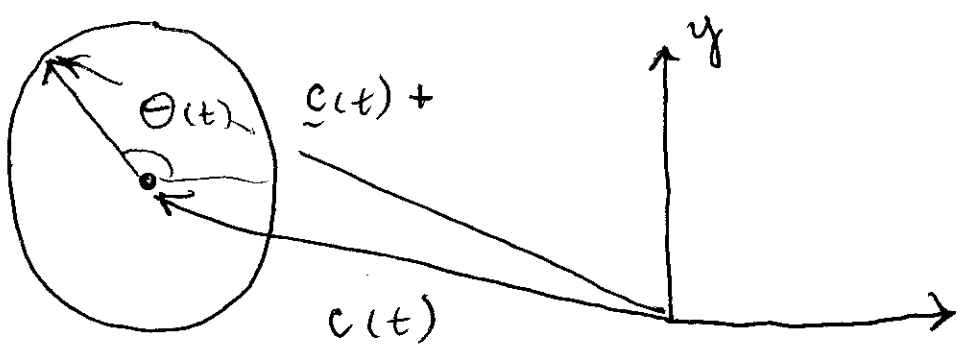
When $t = t_0$, the wheel has done one complete revolution



$$\theta(t_0) = \frac{2\pi t}{t_0}$$

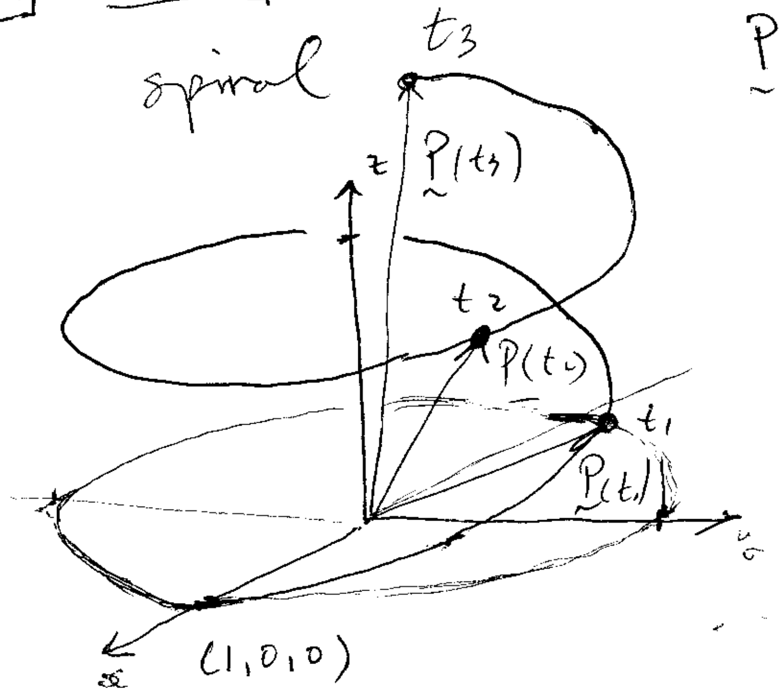
$$(r\cos(\theta(t)),\ r\sin(\theta(t)))$$

IIIb



$$\underset{\sim}{P}(t) = \underset{\sim}{c}(t) + (r\cos([illegible]\,2\pi t),\ r\sin(2\pi t))$$

nb Example



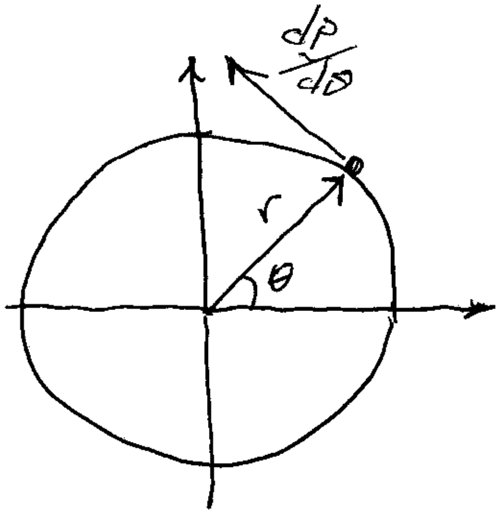
$$\underset{\sim}{P}(t) = (\cos(t), \sin(t), t)$$

$$t > 0$$

nb Polar coordinates

$$= (x(\theta), y(\theta))$$

$$\underset{\sim}{P}(\theta) = (r\cos(\theta), r\sin(\theta))$$

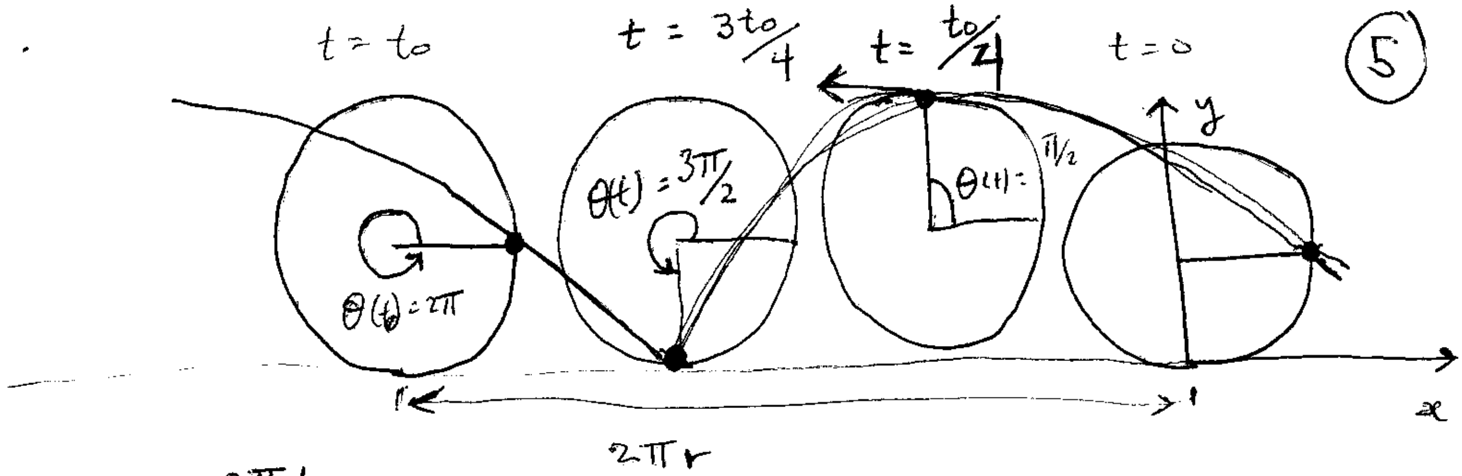


define: $\frac{d\underset{\sim}{P}}{d\theta} = \left(\frac{dx}{d\theta}, \frac{dy}{d\theta} \right)$

$$\frac{d\underset{\sim}{P}}{d\theta} = (-r\sin(\theta), r\cos(\theta))$$

so $\frac{d\underset{\sim}{P}}{d\theta} \cdot \underset{\sim}{P}(\theta) = -r^2\sin(\theta)\cos(\theta) + r^2\cos(\theta)\sin(\theta)$
$= 0$

ib]



$$\theta(t) = \frac{2\pi t}{t_0}$$

$$\underset{\sim}{P}(t) = \underset{\sim}{c}(t) + \left(r\cos\left(\frac{2\pi t}{t_0}\right), r\sin\left(\frac{2\pi t}{t_0}\right)\right)$$

$$= \left(-\frac{t2\pi r}{t_0}, r\right) + \left(r\cos\left(\frac{2\pi t}{t_0}\right), r\sin\left(\frac{2\pi t}{t_0}\right)\right)$$

$$\frac{d\underset{\sim}{P}}{dt} = \left(-\frac{2\pi r}{t_0}, 0\right) + \left(-\frac{r2\pi}{t_0}\sin\left(\frac{2\pi t}{t_0}\right), \frac{r2\pi}{t_0}\cos(\ )\right)$$

ii] so

$$\frac{d\underset{\sim}{P}}{dt}(t_0/4) = \left(-\frac{2\pi r}{t_0}, 0\right) + \left(-\frac{r2\pi}{t_0}, 0\right)$$

$$= \left(-\frac{4\pi r}{t_0}, 0\right)$$

iii]

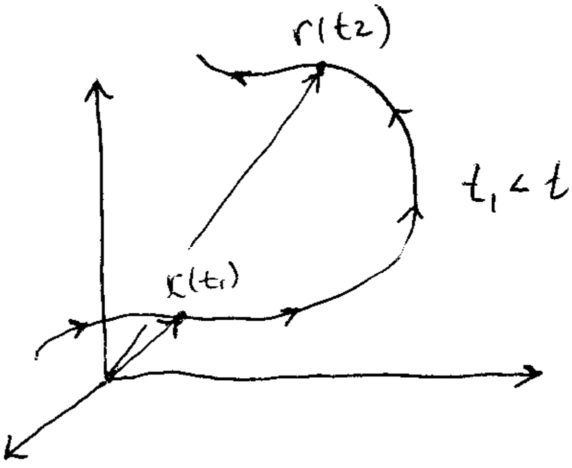
$$\frac{d\underset{\sim}{P}}{dt}(3t_0/4) = \left(-\frac{2\pi r}{t_0}, 0\right) + \left(\frac{r2\pi}{t_0}, 0\right)$$

$$= \underset{\sim}{0}$$

14.1 Vector Functions

10/10/08 ①

1b



- $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$

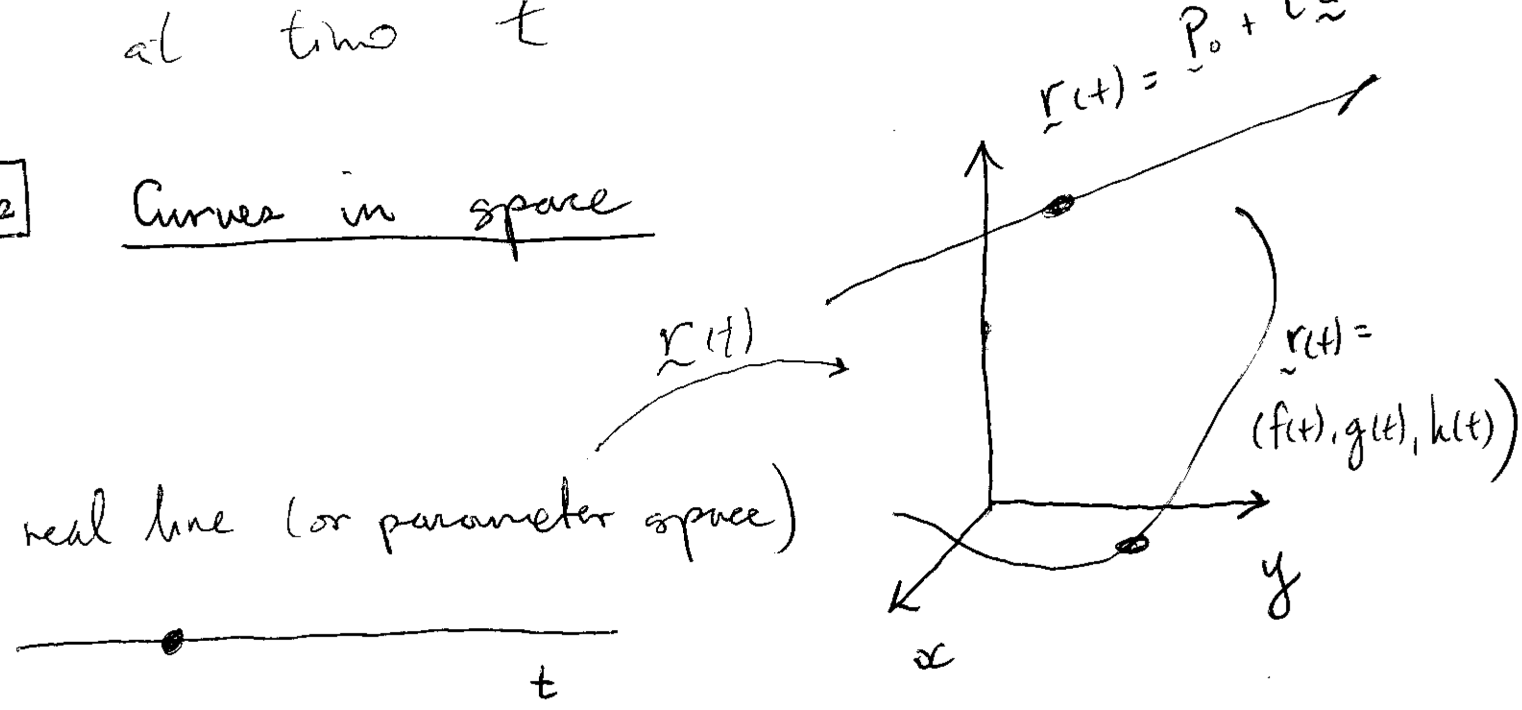
$= (f(t), g(t), h(t))$

- e.g. path of a particle

$\underset{\sim}{r}(t)$ - position vector of the point at time t

- i.e. $\underset{\sim}{r}(t)$ points from the origin to the location of the particle at time t

1b Curves in space

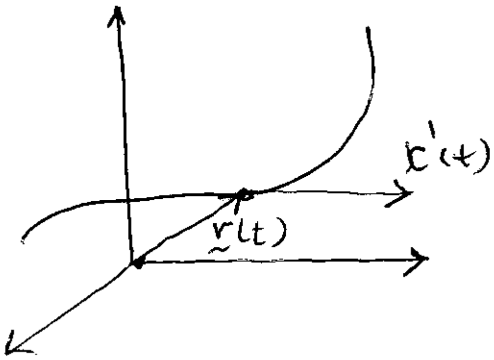


Derivatives of vector valued functions

b

- $\underset{\sim}{r}(t) = (f(t), g(t), h(t))$
- $\frac{d\underset{\sim}{r}}{dt}(t) = (f'(t), g'(t), h'(t))$
- will in general point tangent to the curve given by $\underset{\sim}{r}(t)$

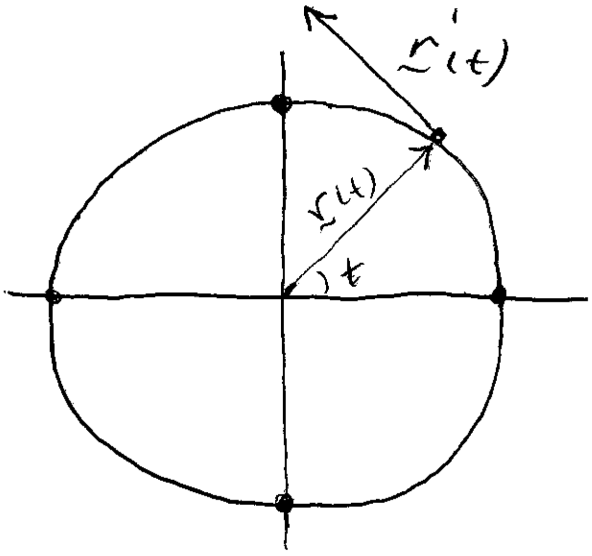
lb



llb E.g.

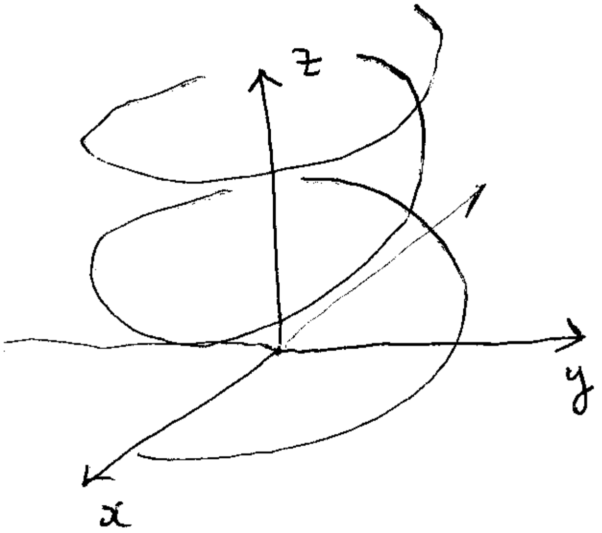
$\underset{\sim}{r}(t) = (\cos(t), \sin(t))$

$\frac{}{t} = \underset{\sim}{r}'(t) = (-\sin(t), \cos(t))$



$\boxed{1b}$ Example

$\underset{\sim}{r}(t) = (\cos(t), \sin(t), t)$ with $t > 0$



– as t increases, particle goes around in a helical pattern

$\boxed{1b}$ Limits and continuity

If $\underline{r}(t) = (f(t), g(t), h(t))$, then

$$\lim_{t\to a} \underset{\sim}{r}(t) = \left(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right)$$

* provided ↗ ↗ ↗ exist

rb] Example ④

$$\underset{\sim}{r}(t) = (1+t^3)\,\underset{\sim}{i} + te^{-t}\,\underset{\sim}{j} + \frac{\sin(t)}{t}\,\underset{\sim}{k}$$

$$\lim_{t \to 0} (1+t^3) = 1$$

$$\lim_{t \to 0} te^{-t} = 0$$

L'hopital's rule

$$\lim_{t \to 0} \frac{\sin(t)}{t} = \frac{\cos(0)}{1} = 1$$

b] $$\therefore \lim_{t \to 0} \underset{\sim}{r}(t) = 1\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{1}\,\underset{\sim}{k}$$

$$= \underset{\sim}{i} + \underset{\sim}{k}$$

$$= (1, 0, 1)$$

ε, δ limits

If $\lim_{t \to a} f(t) = f_a$, then for all $\varepsilon > 0$, there is some [redacted] $\delta(\varepsilon) > 0$ s.t. if $|t-a| < \delta(\varepsilon)$, then $|f(t) - f_a| < \varepsilon$.

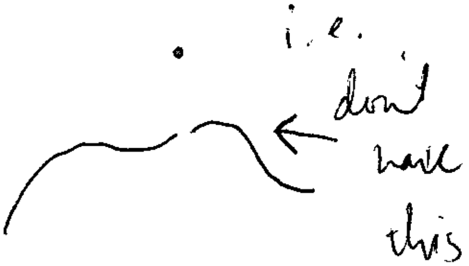
b If $\lim_{t \to a} f(t) = f_a$, $\lim_{t \to a} g(t) = g_a$ and $\lim_{t \to a} h(t) = h_a$, then

$\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. if $|t-a| < \delta(\varepsilon)$

$|t-a| < \delta(\varepsilon)$, then $\| \underset{\sim}{r}(t) - (f_a, g_a, h_a) \| < \varepsilon$

IIb **Continuity**

$\underset{\sim}{r}(t) \overset{(f(t), g(t), h(t))}{=}$ is continuous if



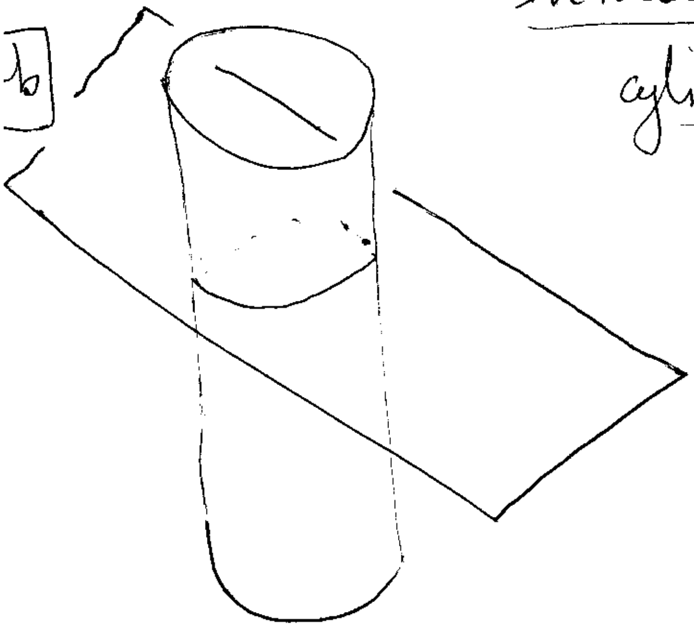
i.e. don't have this

$$\lim_{t \to a} \underset{\sim}{r}(t) = \underset{\sim}{r}(a) = (f(a), g(a), h(a))$$

Example

Intersection of a cylinder and a plane

b



cylinder : $x^2 + y^2 = 1$

$y + z = 2$

6b * we know it's on the cylinder

so $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$ and

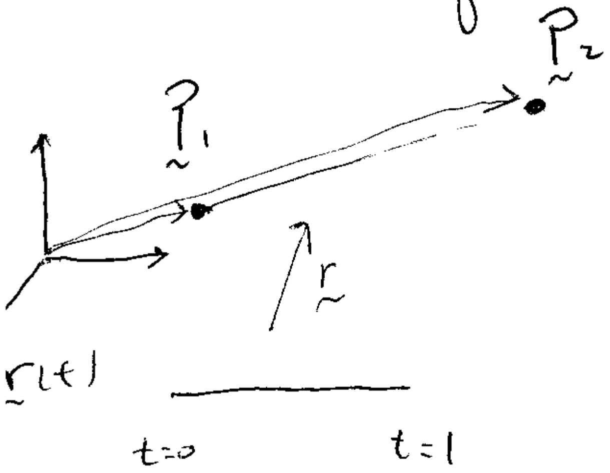
$$x(t) = \cos(t) \quad , \quad \sin(t) = y(t)$$

$$\therefore \quad z(t) = 2 - \sin(t)$$

so $\underset{\sim}{r}(t) = \Big(\cos(t), \sin(t), 2 - \sin(t) \Big)$

Example

vb Line segment connecting two points



w/ $\underset{\sim}{r}(0) = \underset{\sim}{P}_1$

$\underset{\sim}{r}(1) = \underset{\sim}{P}_2$

$$\underset{\sim}{r}(t) = (1-t)\,\underset{\sim}{P}_1 + t\,\underset{\sim}{P}_2$$

$$= \underset{\sim}{P}_1 + t\left(\underset{\sim}{P}_2 - \underset{\sim}{P}_1\right)$$

$\underset{\sim}{r}(0) = \underset{\sim}{P}_1$

$\underset{\sim}{r}(1) = \underset{\sim}{P}_2$

$$\underset{\sim}{r}(1/2) = \underset{\sim}{P}_1 + \frac{1}{2}\left(\underset{\sim}{P}_2 - \underset{\sim}{P}_1\right) = \left(1 - 1/2\right)\underset{\sim}{P}_1 + \frac{1}{2}\underset{\sim}{P}_2$$

$$= \underset{\sim}{P}_1 + \underset{\sim}{P}_2$$

4.2] Derivatives and ~~I~~ integrals of vector valued functions

10/13/08 ①

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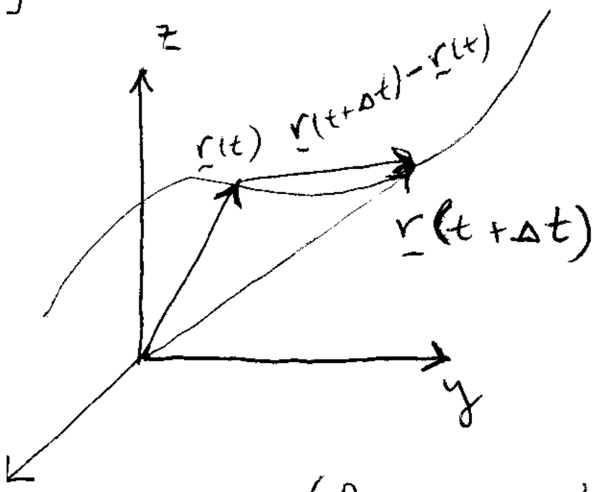


$$-\ \frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{r}'(t) = \lim_{\Delta t \to 0} \frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t}{\Delta t}$$

$$= (f'(t), g'(t), h'(t))$$

$$\underset{\sim}{r}(t) = \left(f(t), g(t), h(t)\right)$$

b] recall: $\lim_{\Delta t \to 0} \underset{\sim}{P}(\Delta t) = \left(\lim_{\Delta t \to 0} P_1(\Delta t), \lim_{\Delta t \to 0} P_2(\Delta t), \lim_{\Delta t \to 0} P_3(\Delta t)\right)$

so if $\underset{\sim}{P}(\Delta t) = \dfrac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t}$

$$= \left(\frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t}{\Delta t} \right)$$

hen

$$\frac{d\underset{\sim}{r}}{dt}(t) = \underset{\sim}{r}'(t) = \left(\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t}{\Delta t} \right)$$

$$= \left(f'(t), g'(t), h'(t) \right)$$

Properties of derivatives of vector valued functions

4]

1. $\frac{d}{dt}[\underset{\sim}{u}(t) + \underset{\sim}{v}(t)] = \underset{\sim}{u}'(t) + \underset{\sim}{v}'(t)$

2. $\frac{d}{dt}[c\,\underset{\sim}{u}(t)] = c\,\underset{\sim}{u}'(t)$

3. $\frac{d}{dt}[f(t)\,\underset{\sim}{u}(t)] = f'(t)\,\underset{\sim}{u}(t) + f(t)\,\underset{\sim}{u}'(t)$

4. $\frac{d}{dt}[\underset{\sim}{u} \cdot \underset{\sim}{v}\,(t)] = \underset{\sim}{u}'(t) \cdot \underset{\sim}{v}(t) + \underset{\sim}{u}(t) \cdot \underset{\sim}{v}'(t)$

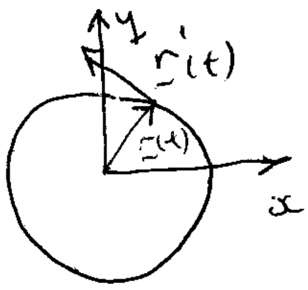
5. $\frac{d}{dt}[\underset{\sim}{u}(t) \times \underset{\sim}{v}(t)] = \underset{\sim}{u}'(t) \times \underset{\sim}{v}(t) + \underset{\sim}{u}(t) \times \underset{\sim}{v}'(t)$

6. $\frac{d}{dt}[\underset{\sim}{u}(f(t))] = f'(t)\,\underset{\sim}{u}'(t)$

Ub]

Example

$$\underset{\sim}{r}(t) \cdot \underset{\sim}{r}'(t) = \frac{1}{2}\left[\underset{\sim}{r}(t) \cdot \underset{\sim}{r}'(t) + \underset{\sim}{r}'(t) \cdot \underset{\sim}{r}(t)\right]$$



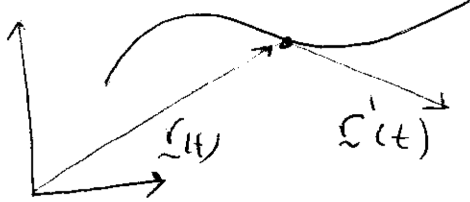
$$= \frac{1}{2}\frac{d}{dt}\,\underset{\sim}{r}(t) \cdot \underset{\sim}{r}(t)$$

$$= \frac{1}{2}\frac{d}{dt}\,|\underset{\sim}{r}|^2$$

?: true for helix?

Tangent vector

$\underset{\sim}{r}'(t)$



C = curve defined by $\underset{\sim}{r}$

Definition: The tangent line to C at a point $\underset{\sim}{P}$ (w/ position vector $\underset{\sim}{r}(t)$) passes through $\underset{\sim}{P}$ and is parallel to $\underset{\sim}{r}'(t)$.

E.g. parameterization of the line

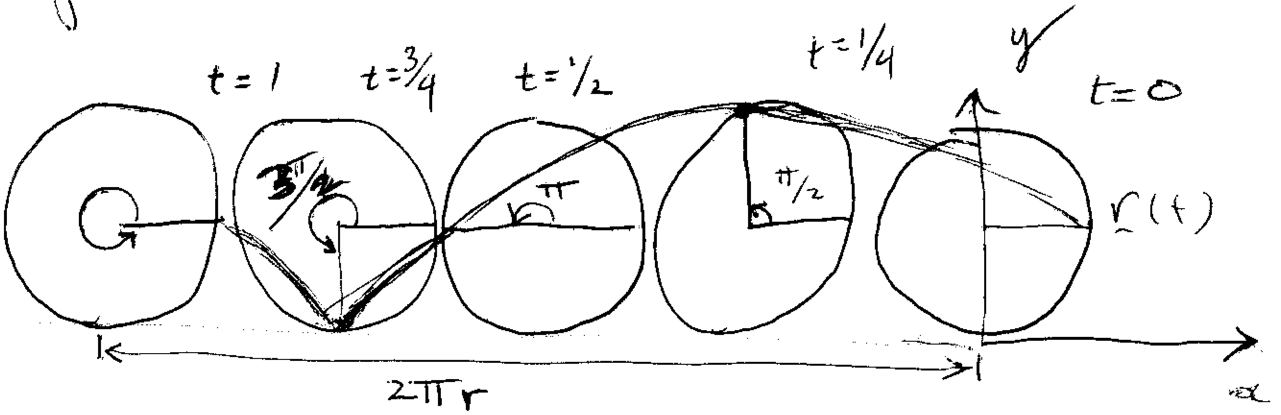
$$\underset{\sim}{l}(s) = \underset{\sim}{r}(t) + s\underset{\sim}{r}'(t)$$

unit tangent vector : $\underset{\sim}{T}(t) = \dfrac{\underset{\sim}{r}'(t)}{|\underset{\sim}{r}(t)|}$

what if $|\underset{\sim}{r}'(t)| = \underset{\sim}{0}$?

- no tangent line
- curve is not "smooth"

E.g. Cycloid



16 $\underset{\sim}{r}(t) = (-2\pi r t, r) + r(\cos(2\pi t), \sin(2\pi t))$

$$\underset{\sim}{r}'(t) = (-2\pi r, 0) + r(-2\pi \sin(2\pi t), 2\pi\cos(2\pi t))$$

$$= 2\pi r(-1 - \sin(2\pi t), \cos(2\pi t))$$

so, $\underset{\sim}{r}'(t) = 0$ if $t = \frac{3}{4}$

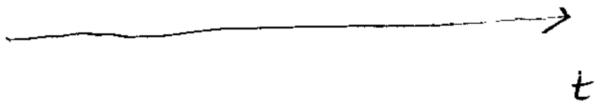
b] Definite integral of continuous function $\underset{\sim}{r}(t)$.

$$\int_a^b \underset{\sim}{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \underset{\sim}{r}(t_i^*)\,\Delta t$$

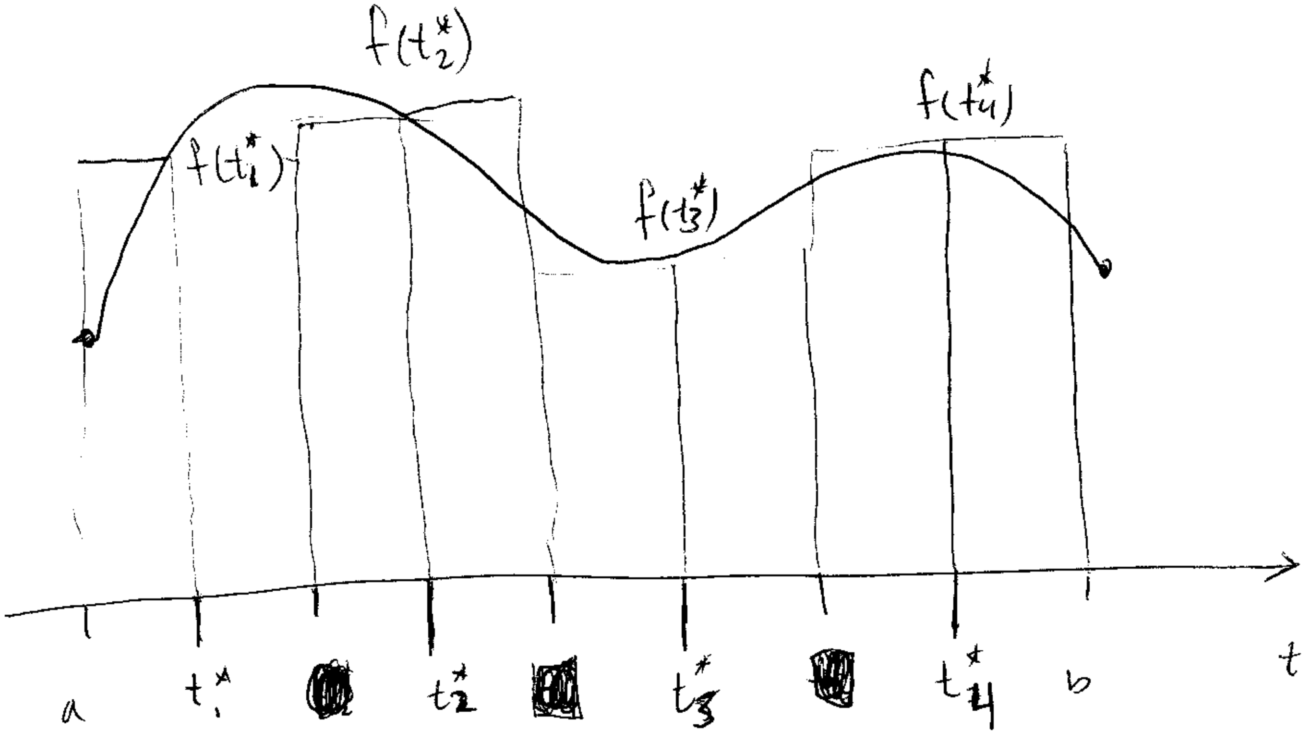
$$= \left(\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\,\Delta t,\ \lim_{n\to\infty} \sum_{i=1}^{n} g(t_i^*)\,\Delta t,\ \lim_{n\to\infty} \sum_{i=1}^{n} h(t_i)\,\Delta t\right)$$

$$= \left(\int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt\right)$$

b]



$$t_i^* = a + \frac{i\,\Delta t}{2}, \quad \Delta t = \frac{b-a}{n}$$



recall :

1] if $F'(t) = f(t)$, then

$$f(t_i^*)\,\Delta t \approx F\left(t_i^* + \frac{\Delta t}{2}\right) - F\left(t_i^* - \frac{\Delta t}{2}\right)$$

so $$\sum_{i=1}^{n} f(t_i^*)\,\Delta t \approx F\left(t_i + \frac{\Delta t}{2}\right) - F(a)$$

$$+ F\left(t_2^* + \frac{\Delta t}{2}\right) - F\left(t_2^* - \frac{\Delta t}{2}\right)$$

$$+ \quad \vdots$$

i.e. $\int_a^b f(t)\,dt = F(t)$ $$+ F(b) - F\left(t_n^* - \frac{\Delta t}{2}\right)$$

$$= F(b) - F(a)$$

x2] and so, if $G'(t) = g(t)$ and

$H'(t) = h(t)$ then

$$\int_a^b \underset{\sim}{r}(t)\,dt = \left(\int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt\right)$$

$$= \big(F(b) - F(a),\ G(b) - G(a),\ H(b) - H(a)\big)$$

$$= \underset{\sim}{R}(b) - \underset{\sim}{R}(a)$$, where $\underset{\sim}{R}(t) = (F(t), G(t), H(t)$

Indefinite integral (or antiderivative)

$$\underset{\sim}{R}(t) = \int \underset{\sim}{r}(t)\, dt$$

Example:

$$\underset{\sim}{r}(t) = 2\cos(t)\, \underset{\sim}{i} + \sin(t)\, \underset{\sim}{j} + 2t\, \underset{\sim}{k}$$

$$\int 2\cos(t)\, dt = 2\sin(t) + C_1$$

$$\int \sin(t)\, dt = -\cos(t) + C_2$$

$$\int 2t\, dt = t^2 + C_3$$

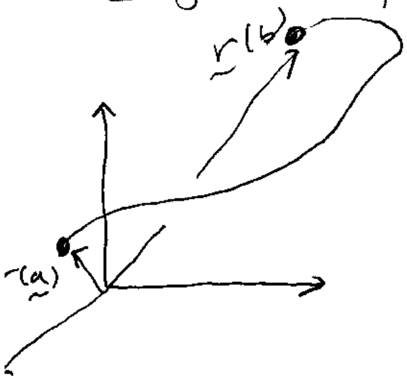
So then $\underset{\sim}{R}(t) = \int \underset{\sim}{r}(t)\, dt = (2\sin(t), -\cos(t), t^2) + (C_1, C_2, C_3)$

$$= (2\sin(t), -\cos(t), t^2) + \underset{\sim}{C}$$

14.3: Arc length and curvature 10/15/08 ①

b]

Length of a curve:



$$L = \int_a^b |\underset{\sim}{r}'(t)|\, dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

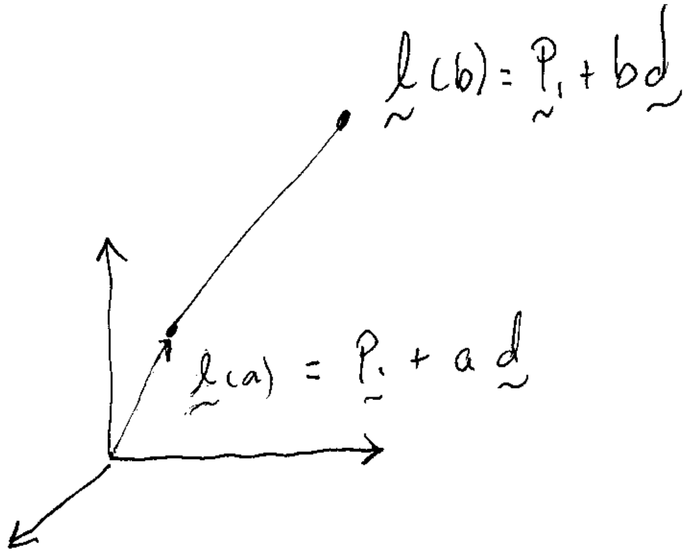
$\underset{\sim}{r}(t) = (f(t), g(t), h(t))$

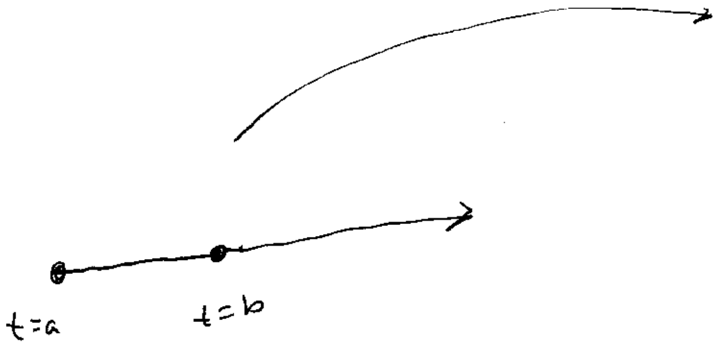
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

ds

Example: Line

$$\underset{\sim}{l}(t) = \underset{\sim}{P_1} + t\,\underset{\sim}{d}$$

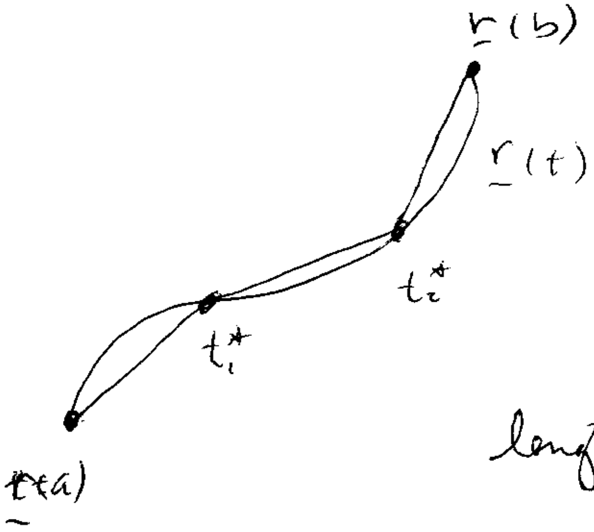




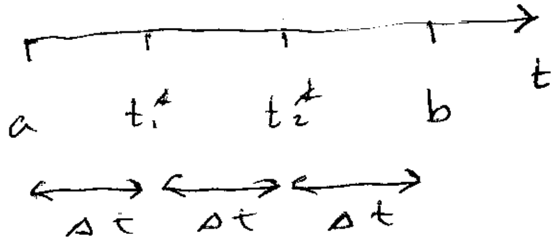
$$\text{length} = \left| \underset{\sim}{P_1} + b\underset{\sim}{d} - (\underset{\sim}{P_2} + a\,\underset{\sim}{d}) \right|$$

$$= (b-a)\,|\underset{\sim}{d}|$$

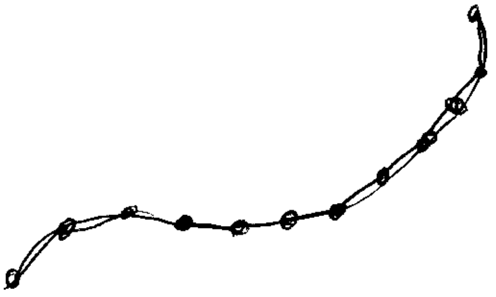
1b



$$\text{length} \approx |\underset{\sim}{r}(t_1^*) - \underset{\sim}{r}(a)|$$
$$+ |\underset{\sim}{r}(t_2^*) - \underset{\sim}{r}(t_1^*)|$$
$$+ |\underset{\sim}{r}(\underset{\sim}{b}) - \underset{\sim}{r}(t_2^*)|$$

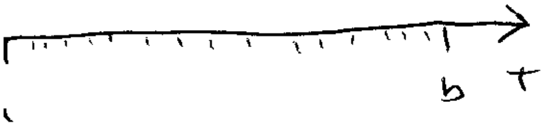


1b even better



$$\text{length} \approx \sum_{i=1}^{n} |\underset{\sim}{r}(t_i^*) - \underset{\sim}{r}(t_{i-1}^*)|$$

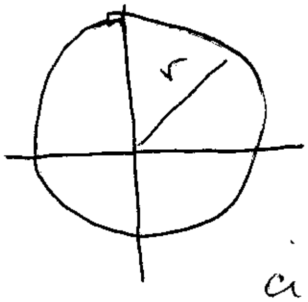
$$= \sum_{i=1}^{n} \left| \frac{\underset{\sim}{r}(t_i^*) - \underset{\sim}{r}(t_{i-1}^*)}{\Delta t} \right| \Delta t$$



$$\text{length} = \lim_{n \to \infty} \sum_{i=1}^{n} \quad ''$$

$$= \int_a^b |\underset{\sim}{r}'(t)| \, dt$$

Example



$$\underset{\sim}{r}(t) = \left(r\cos(2\pi t),\ r\sin(2\pi t)\right)$$

circumference $= \int_0^1 |\underset{\sim}{r}'(t)|\, dt$

$$= \int_0^1 \left| \left(-r2\pi\sin(2\pi t),\ r2\pi\cos(2\pi t)\right) \right| dt$$

b

$$= \int_0^1 \sqrt{r^2(2\pi)^2\sin^2(2\pi t) + r^2(2\pi)^2\cos^2(2\pi t)}\, dt$$

$$= \int_0^1 2\pi r\sqrt{\sin^2 + \cos^2}\, dt$$

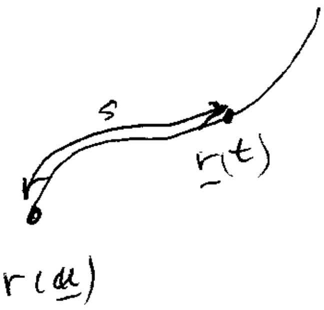
$$= \int_0^1 2\pi r\, dt$$

$$= 2\pi r \int_0^1 dt$$

$$= 2\pi r$$ ✓

Arc length function

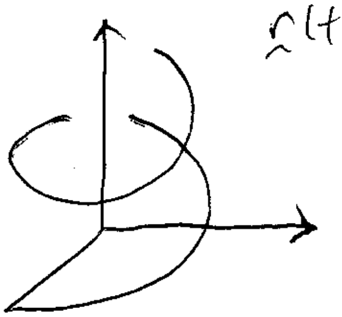
$$s(t) = \int_a^t |\underset{\sim}{r}'(t)|\,dt$$



* can be useful to parameteriz w/ s : $\underset{\sim}{r}(s)$

* however, need to know $t(s)$ to do it

b] Example : helix



$$\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + t\,\underset{\sim}{k}$$

$$\underset{\sim}{r}'(t) = -\sin(t)\,\underline{i} + \cos(t)\,\underline{j} + \underline{k}$$

$$|\underset{\sim}{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

now, $\frac{ds}{dt} = |r'(t)|$ so $\frac{ds}{dt}(t) = \sqrt{2}$

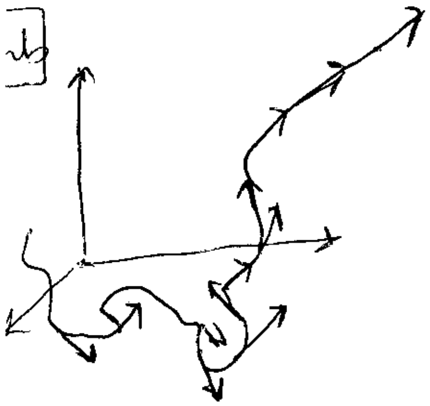
$$\therefore \quad s(t) = \sqrt{2}\,t + s(0) = \sqrt{2}\,t$$

$\therefore \quad t(s) = \frac{s}{\sqrt{2}}$ so $\underline{r}(s) = \underline{r}(t(s))$

$$= \cos(s/\sqrt{2})\,\underset{\sim}{i} + \sin(s/\sqrt{2})\,\underline{j} + \frac{s}{\sqrt{2}}\,\underline{k}$$

Curvature

lb



recall : $\underset{\sim}{T}(t) = \dfrac{r'(t)}{|r'(t)|}$

* curve wiggles alot when $\underset{\sim}{T}$ is changing a lot

ll1 Definition : Curvature

$$k = \left|\frac{d\underset{\sim}{T}}{ds}\right| = \left|\frac{d\underset{\sim}{T}}{dt}\;\frac{dt}{ds}\right| \quad \left(\text{since } \frac{dt}{ds} = 1\Big/\frac{ds}{dt}\right.$$

$$= \left|\frac{d\underset{\sim}{T}}{dt}\Big/ ds/dt\right| = \left|\frac{d\underset{\sim}{T}}{dt}\right|\Big/\left|ds/dt\right|$$

ll2 recall

$$s(t) = \int_a^t |r'(t)|\,dt$$

$$\frac{ds}{dt} = |r'(t)|$$

so

$$k = \left|\frac{d\underset{\sim}{T}}{dt}\right|\Big/|\underset{\sim}{r}'(t)|$$

Normal vector

$\boxed{\text{nb}}$ recall : unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

and $|\mathbf{T}(t)| = 1$

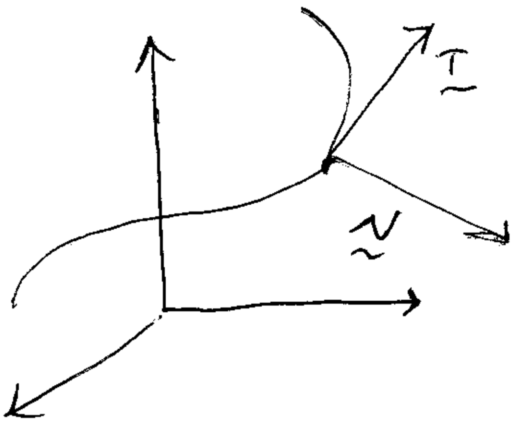
but $|\mathbf{T}(t)|^2 = \mathbf{T}(t) \cdot \mathbf{T}(t) = 1$

$\boxed{\text{lb}}$ so $\dfrac{d}{dt}|\mathbf{T}(t)|^2 = 2\mathbf{T}'(t) \cdot \mathbf{T}(t)$

$= 0$

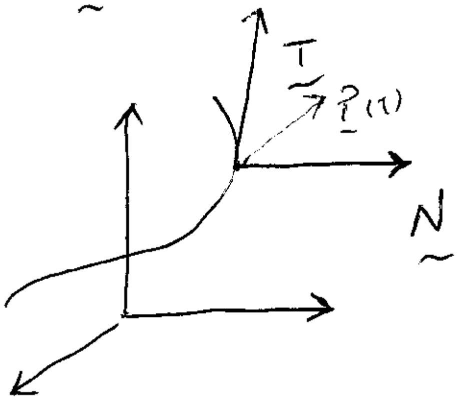
$\therefore$ $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$

unit normal vector : $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$



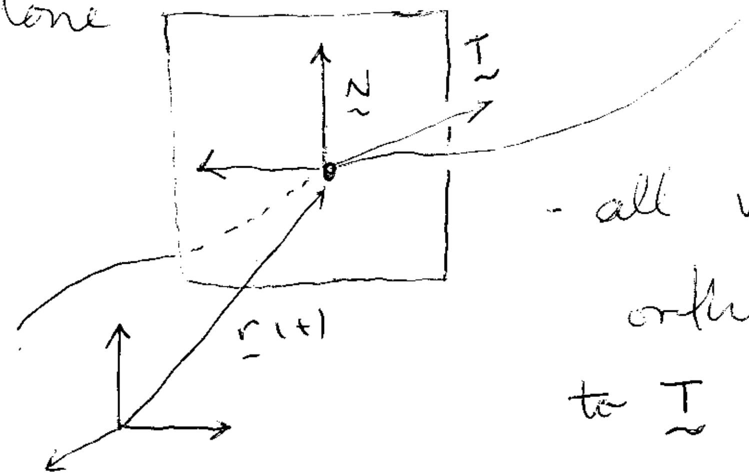
Binormal vector

A $\underset{\sim}{P}(t) = \underline{T}(t) \times \underset{\sim}{N}(t)$



* $\underline{P}$ and $\underset{\sim}{N}$ define a plane locally orthogonal to the curve

b Normal plane



- all vectors orthogonal to $\underset{\sim}{T}$

llb Osculating plane

- vectors orthogonal to $\underset{\sim}{B}$
  - $\underset{\sim}{T}$ and $\underset{\sim}{N}$ in the plane
- plane that the curve most nearly fits in at $\underset{\sim}{r}(t)$

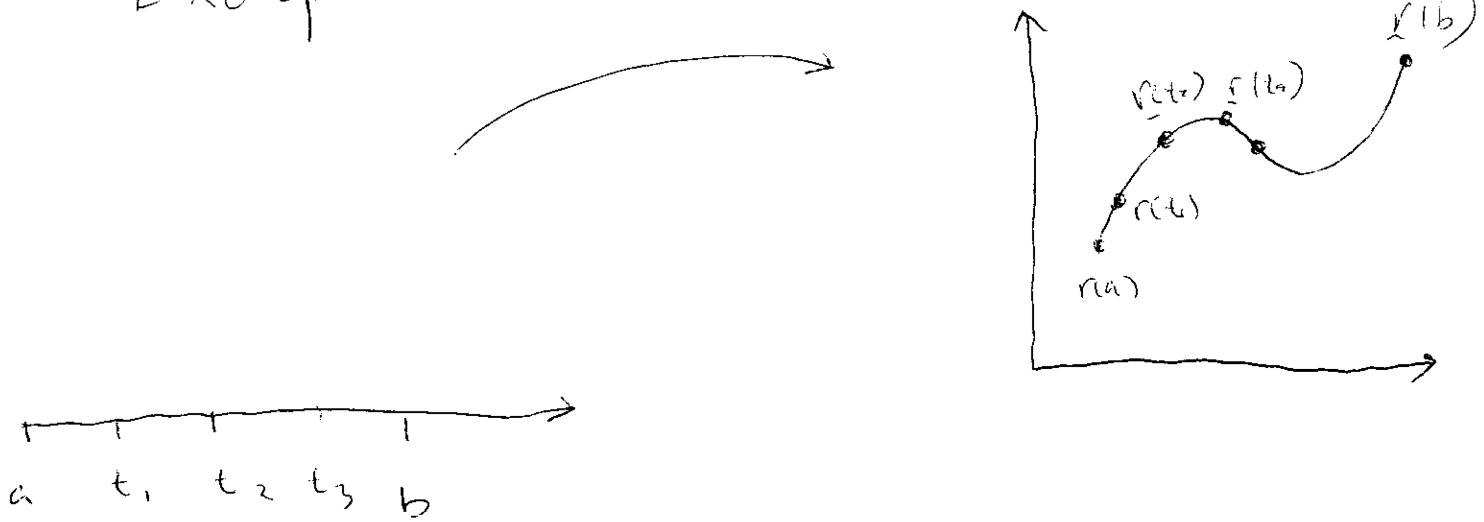
Arc length

10/17/08

rb - parameterization w.r.t. arc length

- removes effect of distortion
- no stretching or squishing

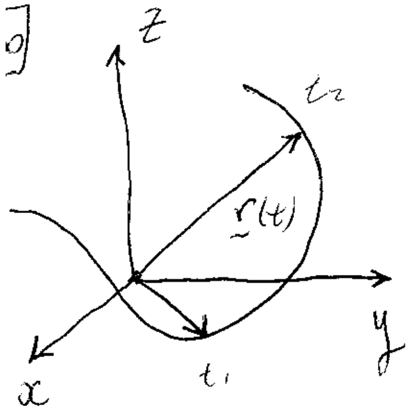
rb Example



Rlb

4.4 Motion in space

10/17/08 ①



- $\underset{\sim}{r}(t)$: position of a particle in space at time t

- $\underset{\sim}{r}'(t) = \underset{\sim}{v}(t)$ velocity of the particle at time t

$'(t) = |\underset{\sim}{v}(t)|$ = speed of particle at time t

$\dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}(t)|}$ = instantaneous direction of the particle

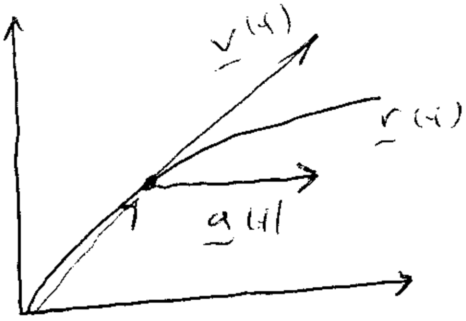
$\underset{\sim}{a}(t) = \underset{\sim}{v}'(t) = \underset{\sim}{r}''(t)$: acceleration of the particle

Example

$\underset{\sim}{r}(t) = t^3 \underset{\sim}{i} + t^2 \underset{\sim}{j}$

$\underset{\sim}{v}(t) = 3t^2 \underset{\sim}{i} + 2t \underset{\sim}{j}$

$\underset{\sim}{a}(t) = 6t \underset{\sim}{i} + 2 \underset{\sim}{j}$



Newton's second law of motion

Ib

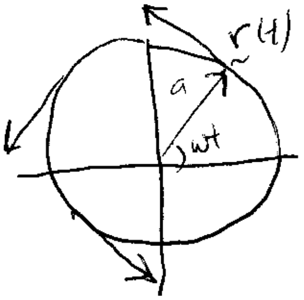
$$\underset{\sim}{F}(t) = m\,\underset{\sim}{a}(t)$$

force acting on the particle

particle mass

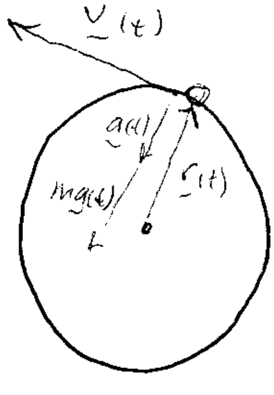
IIb **Example**: What force maintains a circular orbit?

$$\underset{\sim}{r}(t) = a\cos(\omega t)\,\underset{\sim}{i} + a\sin(\omega t)\,\underset{\sim}{j}$$



$$\underset{\sim}{v}(t) = -a\omega\sin(\omega t)\,\underset{\sim}{i} + a\omega\cos(\omega t)\,\underset{\sim}{j}$$

$$\underset{\sim}{a}(t) = -a\omega^2\cos(\omega t)\,\underset{\sim}{i} - a\omega^2\sin(\omega t)\,\underset{\sim}{j}$$

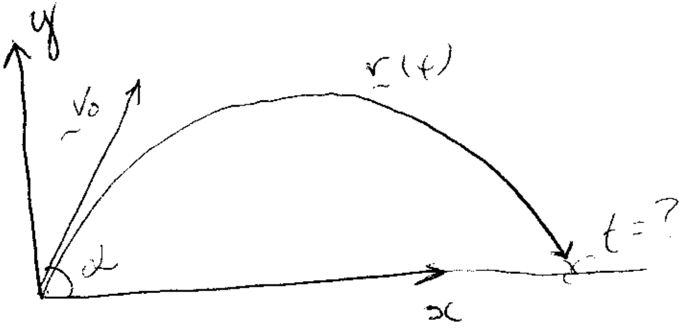


So $\underset{\sim}{F} = m\,\underset{\sim}{a}(t)$

$$= -ma\omega^2\cos(\omega t)\,\underset{\sim}{i} - ma\omega^2\sin(\omega t)$$

Example : Projectile trajectory

b)



$$\underset{\sim}{F} = m\underset{\sim}{a} = -mg\underset{\sim}{j}$$

$$\underset{\sim}{a} = -g\underset{\sim}{j} \quad \rightarrow \quad \underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$$

$$= -gt\underset{\sim}{j} + \underset{\sim}{c_1}$$

$$\underset{\sim}{v_0} = |\underset{\sim}{v_0}|(\cos(\alpha), \sin(\alpha))$$

$$\rightarrow \quad \underset{\sim}{x}(t) = \int \underset{\sim}{v}(t)\,dt$$

$$= -\frac{gt^2}{2}\underset{\sim}{j} + \underset{\sim}{c_1}t + \underset{\sim}{c_0}$$

b)

$$\underset{\sim}{r}(0) = \underset{\sim}{0} \quad \rightarrow \quad \underset{\sim}{c_0} = \underset{\sim}{0}$$

$$\underset{\sim}{v}(0) = \underset{\sim}{v_0} \quad \rightarrow \quad \underset{\sim}{x}(t) = -\frac{gt^2}{2}\underset{\sim}{j} + \underset{\sim}{v_0}t$$

$$\underset{\sim}{c_1} = \underset{\sim}{v_0}$$

when does it hit the ground

$$y(t) = -\frac{gt^2}{2} + |\underset{\sim}{v_0}|\sin(\alpha)t = 0$$

$$\therefore\ t = 0 \quad \text{and} \quad t = \frac{2|\underset{\sim}{v_0}|\sin(\alpha)}{g}$$

Tangential and normal components of acceleration

b]

unit tangent : $\underset{\sim}{T}(t) = \dfrac{\underline{r}'(t)}{|r'(t)|} = \dfrac{\underline{v}(t)}{v(t)}$

i.e. $\underline{v}(t) = v(t)\,\underset{\sim}{T}(t)$

$\therefore\ \underset{\sim}{a}(t) = v'(t) = v'(t)\,\underset{\sim}{T}(t) + v(t)\,\underset{\sim}{T}'(t)$

a] recall

$K = \dfrac{|\underline{T}'|}{|\underset{\sim}{r}'|} = \dfrac{|\underset{\sim}{T}'|}{v}$ i.e. $kv = |\underset{\sim}{T}'|$

$\underset{\sim}{N}(t) = \dfrac{\underset{\sim}{T}'(t)}{|\underset{\sim}{T}'(t)|} \rightarrow \underset{\sim}{T}'(t) = kv\underset{\sim}{N}(t)$

$\therefore\ \underset{\sim}{a}(t) = v'(t)\,\underset{\sim}{T}(t) + kv^2\,\underset{\sim}{N}(t)$

note * $v'(t) = \dfrac{\underset{\sim}{r}'(t)\cdot\underset{\sim}{r}''(t)}{|\underset{\sim}{r}'(t)|}$, $kv^2 = \left|\dfrac{\underset{\sim}{r}'(t)\times\underset{\sim}{r}''(t)}{|\underset{\sim}{r}'(t)|}\right|$