9/26/08

Math 32A: Multivariable Calculus

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Office Hours: MW 9-10 AM (MS 7619E)

Textbook: Multivariable Calculus, Stewart
(sixth edition)

Course Webpage:

http://www.math.ucla.edu/~jteran/32a.1.08f/

- syllabus
- tentative lecture schedule
  - e.g. 9/26 : 13.1
- exam schedule
  - Midterm 1: 10/24
  - Midterm 2: 11/14

- Final Exam: 12/9 (11:30 AM - 2:30 PM)

Grading ~~Policy~~

15% = Homework

40% = 2 Midterms

- highest counted as 25%

lowest as 15%

45% = Final Exam

Policy

1. HW due every monday at the end of class (starting 10/6)

* no late HW

2. No make up exams

- missed exam counts as a zero

Homework Details

1. Only use the front side of the paper

2. Write your name and section (1a - 1f) and ID number on the back of the last page.

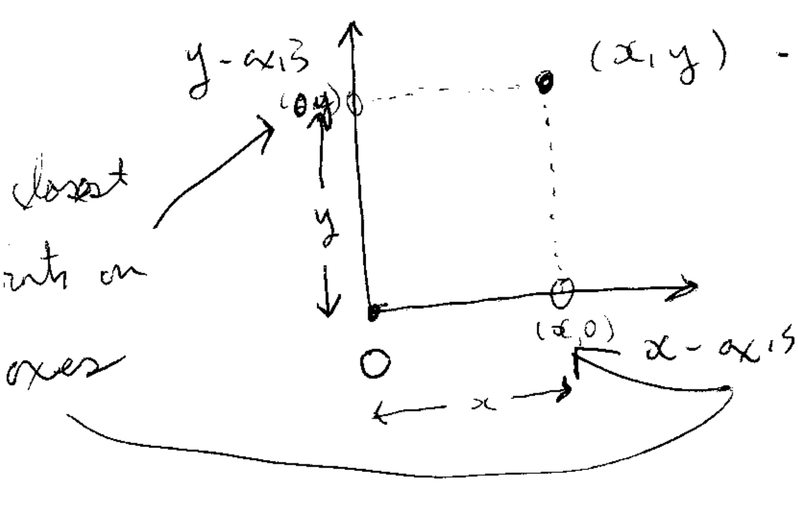
3. Homework will be graded on a 10 point scale. Three problems (at random) will be graded for two points each, 4 points given for completeness of work

Multivariable Functions

- used throughout physics + engineering
- defined over multiple dependent variables
- e.g. $f(x, y, z, t)$
  - density
  - temperature
  - fluid/air velocity
  - pressure

Coordinate Systems

- two dimensional (planar)

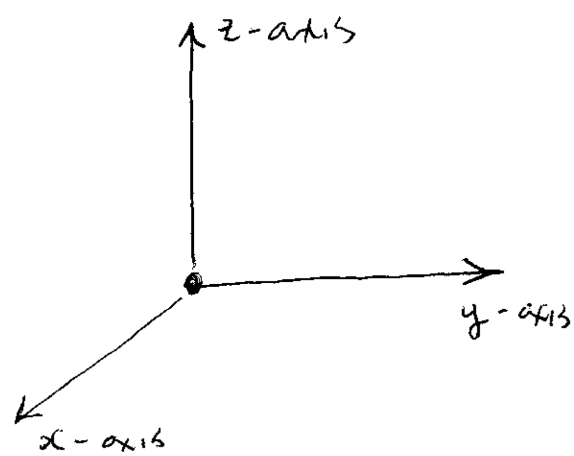


- point in the plane relative to origin O and ~~axes~~ x, y axes

e.g. non-ortogonal

Three Dimensional Coordinate Systems

- point in space $= (x, y, z)$
- ordred triple
  - i.e. order matters
    $(1, 5, 3) \neq (3, 1, 5)$
- (x, y, z) relative to an
- origin (O) and
- coordinate axis
  - x - axis
  - y - axis
  - z - axis



Axis Orientation

- right-handed system
- z-axis determined from x-axis + y-axis by "right-hand rule"
- (i) point thumb in direction of z-axis

(ii) align fingers w/ x-axis

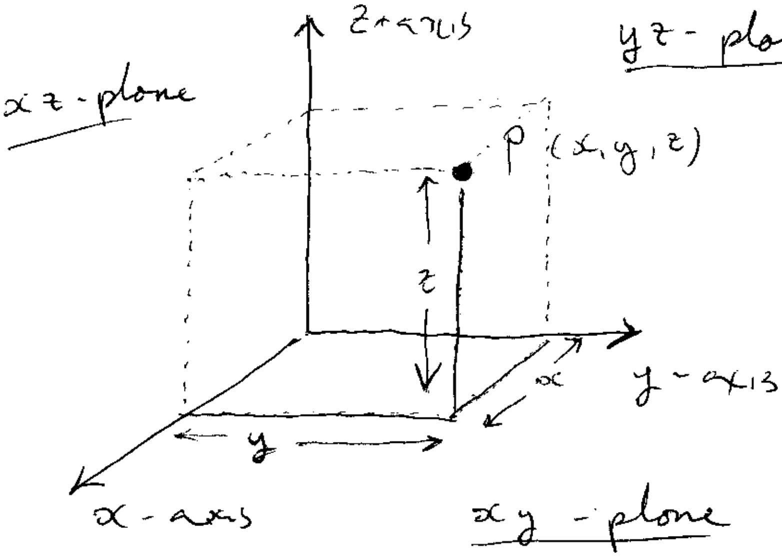
(iii) y-axis should be approached by curling fingers counterclockwise

* don't use your left-hand

Points

P - (x, y, z)

- relative to origin + coordinate axes



x - distance from P to yz-plane

y - " " " " xz-plane

z - " " " " xy-plane

Q: What is (x, y, z) of the origin?

What is the (x, y, z) of a point on an axis?

Distance between points

Definition: Given points $P_1 = (x_1, y_1, z_1)$

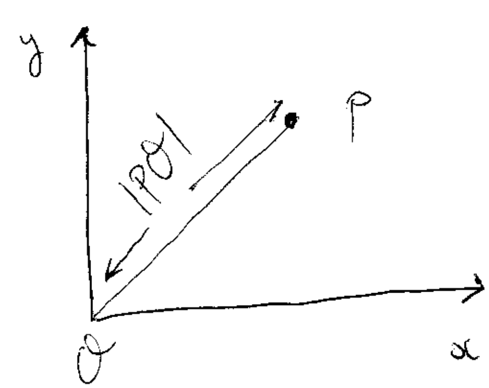
$+ \; P_2 = (x_2, y_2, z_2)$

the distance between the points is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

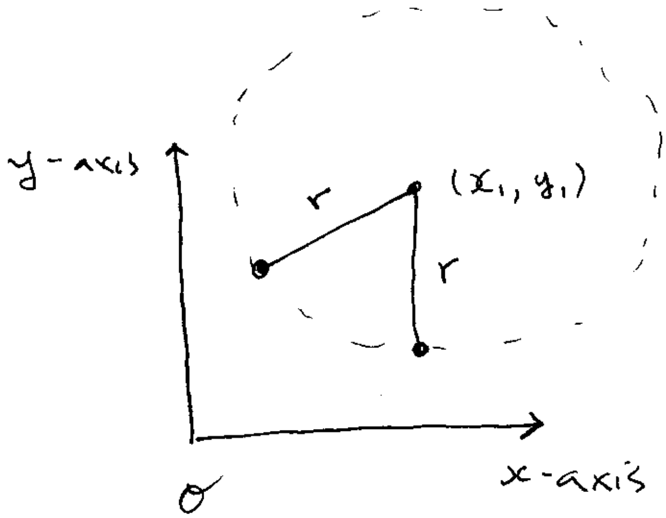
E.g. Distance from a point in the xy-plane ($P = (x, y, 0)$) to the origin is

$$|PO| = \sqrt{x^2 + y^2}$$

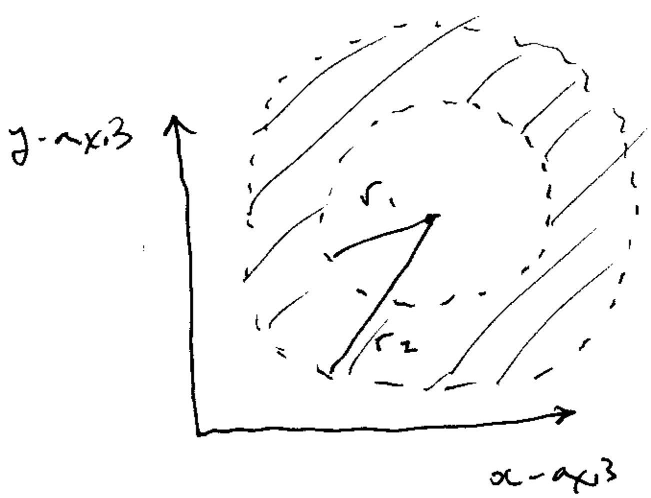


E.g. $\sqrt{(x-x_1)^2 + (y-y_1)^2} = r$

and $\leq r$



E.g. $r_1 \leq \sqrt{(x-x_1)^2 + (y-y_1)^2} \leq r_2$



Vectors

9/29/08 (1)

b - quantity w/ direction and magnitude

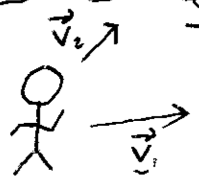
- typically represented as an arrow

e.g.

- notation: $\vec{v}$ or $\underset{\sim}{v}$

bold face

-b - examples $(\vec{v})$

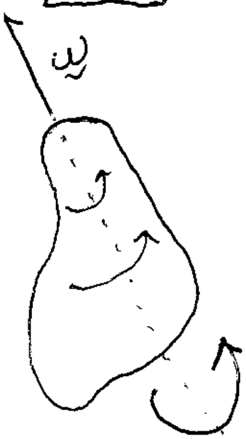


- velocity: how fast and in what direction

- force: how hard are you pulling + in which direction

$\vec{f}$

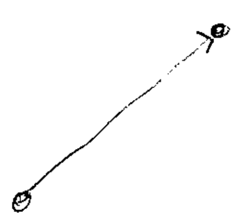
$\underset{\sim}{\omega}$



- angular velocity: in which plane are you spinning + at how many rotations/s

$\vec{\omega}$, $\underset{\sim}{\omega}$

rrb - determined by two points A, B



B

A

$\overrightarrow{AB}$

* note: $\overrightarrow{AB} \neq \overrightarrow{BA}$

rrb cont'd

- notice that $\overrightarrow{AB}$ may be equal to $\overrightarrow{CD}$

* considered to be equal if when placed at the same starting point, they point to the same ending point

lb Vector Addition / Subtraction / ~~Scalor~~ Scalor Multiplication

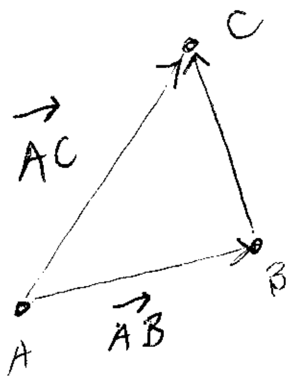
- given $\vec{V}_1$ + $\vec{V}_2$, add $\vec{V}_3 = \vec{V}_1 + \vec{V}_2$

  - subtract $\vec{V}_4 = \vec{V}_1 - \vec{V}_2$

  - scalor multiply $\vec{V}_5 = s\,\vec{V}_1$

    e.g. $\vec{V}_5 = 3.79\,\vec{V}_1$

rb Geometric Equivalents



$\vec{AC} = \vec{AB} + \vec{BC}$

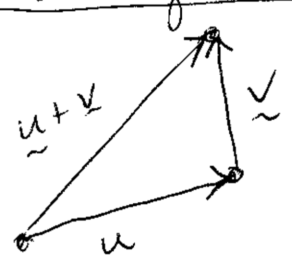
i.e. $\vec{AC}$ net path after going

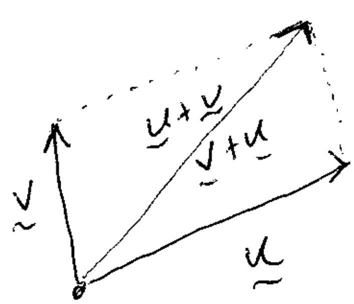
A → B then B → C

(triangle ~~identity~~ law)

Parallelogram Law

tri
aw





$\underset{\sim}{u}+\underset{\sim}{v} =$

$\underset{\sim}{v}+\underset{\sim}{u}$

$\underset{\sim}{v}$'s tail at $\underset{\sim}{u}$'s head, then

connect $\underset{\sim}{u}$'s tail to v's head

Scalar Multiplication

b - given scalar c + vector $\underset{\sim}{v}$

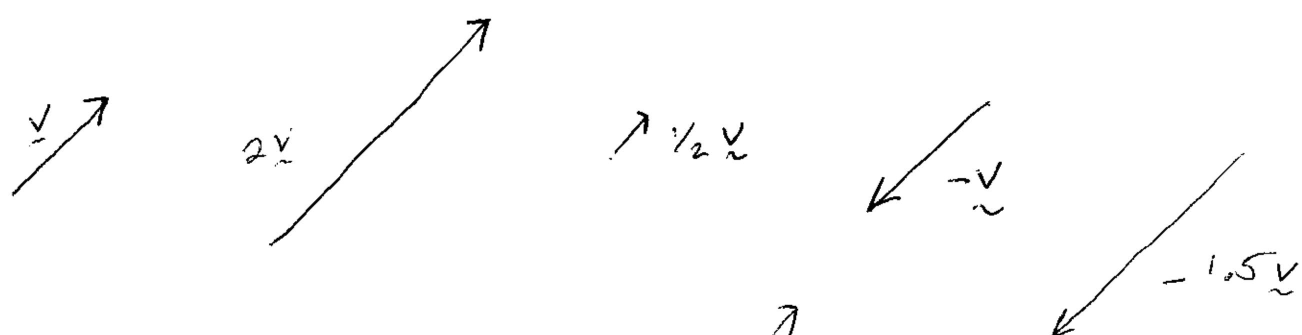
$c\underset{\sim}{v}$ is also a vector

- if $c \neq 0$, $c\underset{\sim}{v}$ is in the same direction as $\underset{\sim}{v}$ w/ magnitude $|c||\underset{\sim}{v}|$

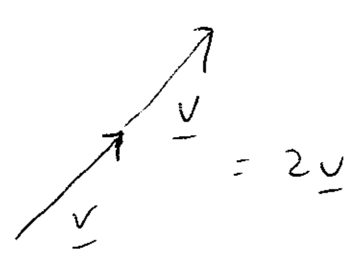
* if c is negative, direction is opposite to $\underset{\sim}{v}$

* $(-1)\underset{\sim}{v} = -\underset{\sim}{v}$

b



* note: $2\underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{v}$

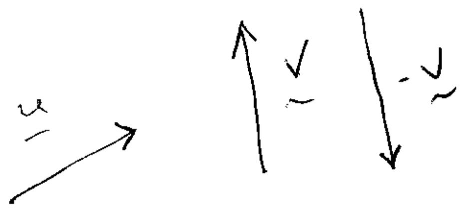


Vector Subtraction

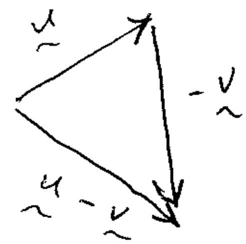
b

$$\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{u} + (-\underset{\sim}{v})$$

triangle law

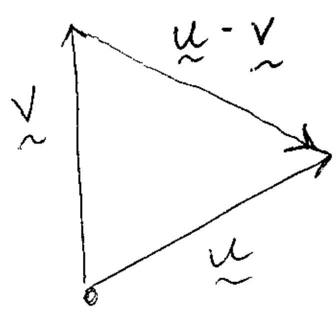


$\underset{\sim}{u} - \underset{\sim}{v} =$



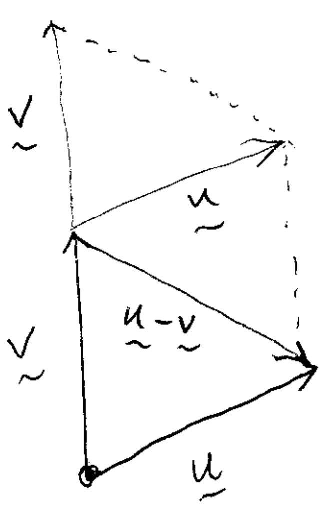
b

parallelogram law (subtraction



- allign tails of $\underset{\sim}{u}$ + $\underset{\sim}{v}$
- connect head of $\underset{\sim}{v}$ to head of $\underset{\sim}{u}$, this is $\underset{\sim}{u} - \underset{\sim}{v}$

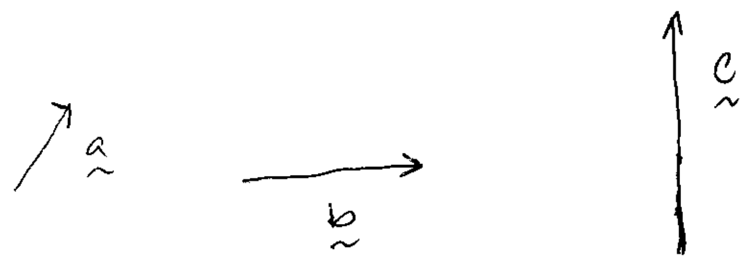
rb

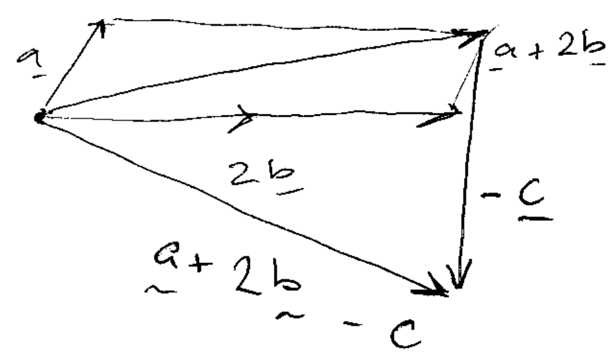


$$(\underset{\sim}{u} - \underset{\sim}{v}) + \underset{\sim}{v} = \underset{\sim}{u}$$

Example :

b] $\underset{\sim}{a} + 2\underset{\sim}{b} - \underset{\sim}{c}$



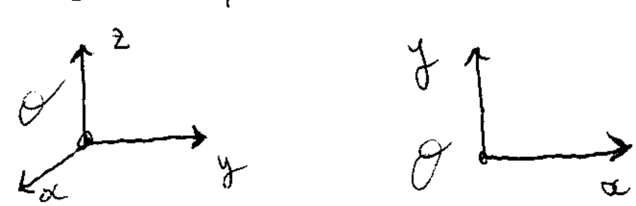


b] **Components**

- introduction of coordinate frame $\longrightarrow$ algebraic treatment of vectors

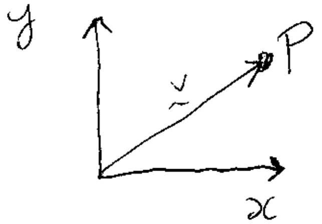
- i.e. $\underset{\sim}{v} = (v_1, v_2, v_3)$

- coord frame = axes + O origin



rrb — components

- place $\underset{\sim}{v}$ w/ tail at $\mathcal{O}$



- $\underset{\sim}{v} = (v_1, v_2, v_3)$

w/ coords of $P = (v_1, v_2, v_3)$

- $\underline{v}$ points from $\mathcal{O}$ to a point P

(v_1, v_2, v_3) are the coordinates of that point

llb Example

Given points A + B, $\vec{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

where

$A = (a_1, a_2, a_3)$

$B = (b_1, b_2, b_3)$

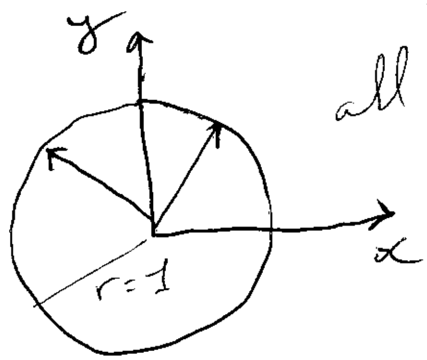
rb Magnitude

Given $\underset{\sim}{v} = (v_1, v_2, v_3)$

$$|\underset{\sim}{v}| = \|\underset{\sim}{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

* consistent w/ distance notion for points

all vectors of length 1



lb Algebraic Addition + Scaling

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$\underset{\sim}{a} - \underset{\sim}{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

$$c\underset{\sim}{a} = (ca_1, ca_2, ca_3)$$

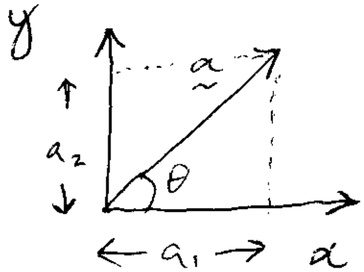
rb

Example

- direction + magnitude + coordinate frame

iff $\underline{a} = (a_1, a_2)$

$|\underline{a}| = a$



$$a_1 = a \cos(\theta)$$

$$a_2 = a \sin(\theta)$$

rrb

Two dimensional, three dimensional + n-dimensional vectors

$\underset{\sim}{a} = (a_1, a_2)$ $\qquad$ $\underline{a} = (a_1, a_2, a_3)$

$$\underline{a} = (a_1, a_2, \ldots, a_n)$$

lb

Standard Basis Vectors

- $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$

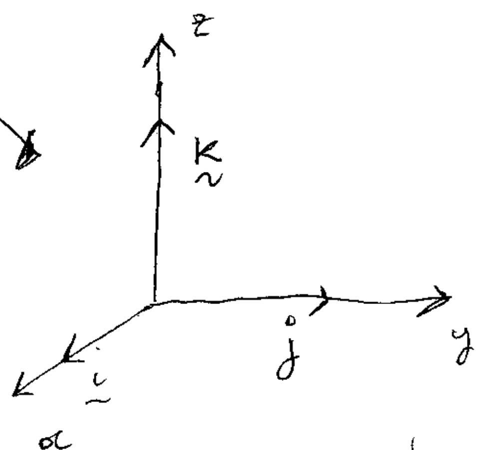
  points along x-axis

  points along y-axis

  points along z-axis

rb

- each of unit length



$$\underset{\sim}{a} = (a_1, a_2, a_3)$$

$$= a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$$

- any vector can be expressed in terms of standard basis vectors

a_i = ith component of $\underset{\sim}{a}$

lb Unit Vectors

Given $\underset{\sim}{a} \neq \underset{\sim}{0}$ (i.e. w/ $|\underset{\sim}{a}| \neq 0$),

$\underset{\sim}{u} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is the unit vector

in the direction of $\underset{\sim}{a}$

rb $|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

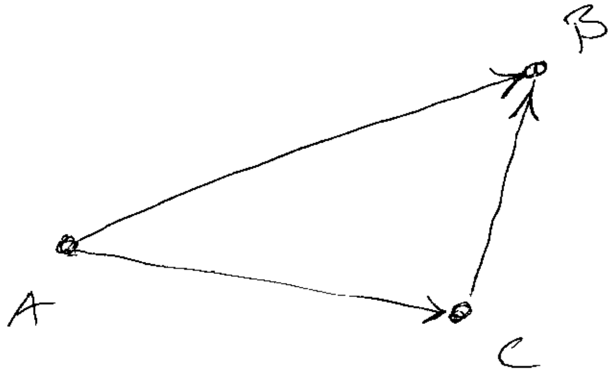
$$\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \left(a_1/|\underset{\sim}{a}|, a_2/|\underset{\sim}{a}|, a_3/|\underset{\sim}{a}| \right)$$

$$\left| \underset{\sim}{a}/|\underset{\sim}{a}| \right| = \sqrt{a_1^2/|\underset{\sim}{a}|^2 + a_2^2/|\underset{\sim}{a}|^2 + a_3^2/|\underset{\sim}{a}|^2}$$

$$= \frac{1}{|\underset{\sim}{a}|} \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$= |\underset{\sim}{a}|/|\underset{\sim}{a}| = 1$$

Triangle Inequality



$$|\vec{AB}| \le |\vec{AC}| + |\vec{CB}|$$

- shortest distance between two points = straight line

The Dot Product

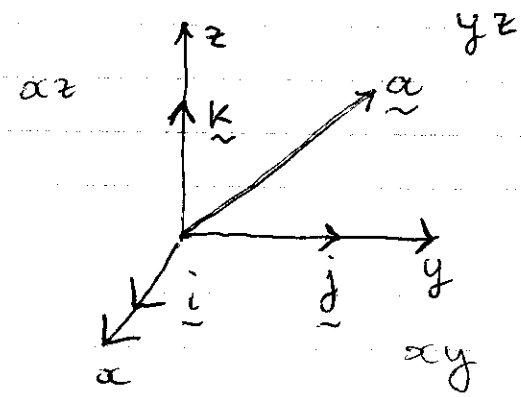
- operations on vectors
  - addition : vector + vector = vector
  - scalar multiplication :

    scalar $*$ vector = vector
  - dot product : vector $\cdot$ vector = scalar

Definition : If $\underset{\sim}{a} = (a_1, a_2, a_3)$ + $\underset{\sim}{b} = (b_1, b_2, b_3)$, then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Example :



$\underset{\sim}{a} \cdot \underset{\sim}{i} = a_1$

$\underset{\sim}{a} \cdot \underset{\sim}{j} = a_2$

$\underset{\sim}{a} \cdot \underset{\sim}{k} = a_3$

$\underset{\sim}{i} = (1, 0, 0)$

$\underset{\sim}{j} = (0, 1, 0)$

$\underset{\sim}{k} = (0, 0, 1)$

i.e. [redacted] geometric interpretation of the dot product

rb

Properties of the dot product

1. $\underset{\sim}{a} \cdot \underset{\sim}{a} = a_1^2 + a_2^2 + a_3^2 = |\underset{\sim}{a}|^2$

$|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

2. $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$= b_1 a_1 + b_2 a_2 + b_3 a_3$

$= \underset{\sim}{b} \cdot \underset{\sim}{a}$ [redacted]

lb

3. $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) =$

$a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$

$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$

$= \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

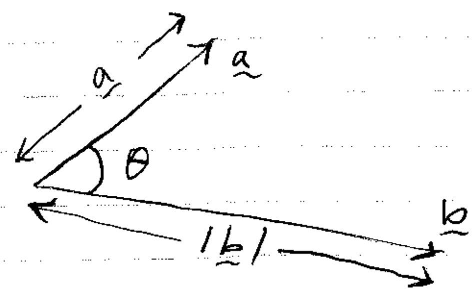
$(c\underset{\sim}{a}) \cdot \underset{\sim}{b} = ca_1 b_1 + ca_2 b_2 + ca_3 b_3$

$c\underset{\sim}{a} = (ca_1, ca_2, ca_3)$ $\quad = c(\underset{\sim}{a} \cdot \underset{\sim}{b})$

Geometric Property of Dot Product

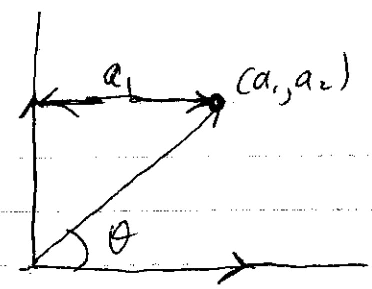
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$$

$0 \leq \theta \leq \pi$



Example:

$\underset{\sim}{a} \cdot \underset{\sim}{i} = a_1$



$a_1 = \cos\theta$

so, $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{b}| \left(\underset{\sim}{a} \cdot \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|} \right) \longrightarrow$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$ for $\underset{\sim}{b} = b\underset{\sim}{i} = (b, 0)$

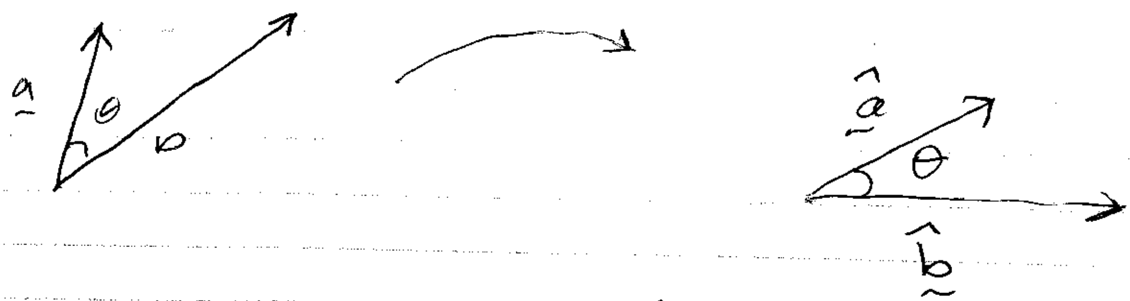
Remark

- can prove that $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$ from definition of $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- e.g. law of cosines
- e.g. linear algebra

4b

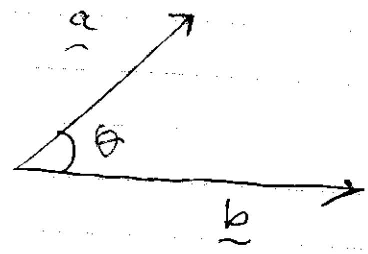
Proof that $a_1 b_1 + a_2 b_2 + a_3 b_3 = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$

- rotation + translation preserve relative lengths and angles



i.e. $\underset{\sim}{a} \cdot \underset{\sim}{b} = \hat{\underset{\sim}{a}} \cdot \hat{\underset{\sim}{b}}$

rb



$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

(for $|\underline{a}| \neq 0$, $|\underline{b}| \neq 0$)

* dot product

measures extent to which $\underline{a}$ and $\underline{b}$ point in the same direction

lb Example:

(i) $\dfrac{\underset{\sim}{i} \cdot \underset{\sim}{j}}{|\underset{\sim}{i}||\underset{\sim}{j}|} = 0 = \cos(\pi/2)$

(ii) $\underset{\sim}{a} = (2, 2, -1)$, $\underset{\sim}{b} = (5, -3, 2)$

$|\underset{\sim}{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$|\underset{\sim}{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 \cdot 5 - 2 \cdot 3 - 1 \cdot 2 = 2$

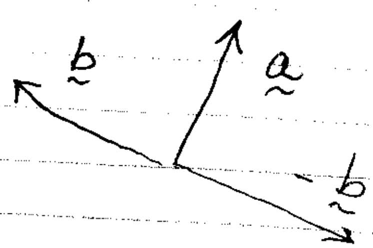
so $\cos(\theta) = \dfrac{2}{3\sqrt{38}}$ i.e. $\theta = \cos^{-1}($ " $)$

<u>Definition</u>: Two vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ are orthogonal if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

* everything is orthogonal to $\underset{\sim}{0}$.

$\boxed{\text{E.g.}}$ (a_1, a_2) how many non-zero vectors $\underset{\sim}{b}$ in the plane are orthogonal to it?

$$\underset{\sim}{b} = (-a_2, a_1) \quad \text{or} \quad \underset{\sim}{b} = (a_2, -a_1)$$



$$c\, \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot (c\underset{\sim}{b}) = 0$$

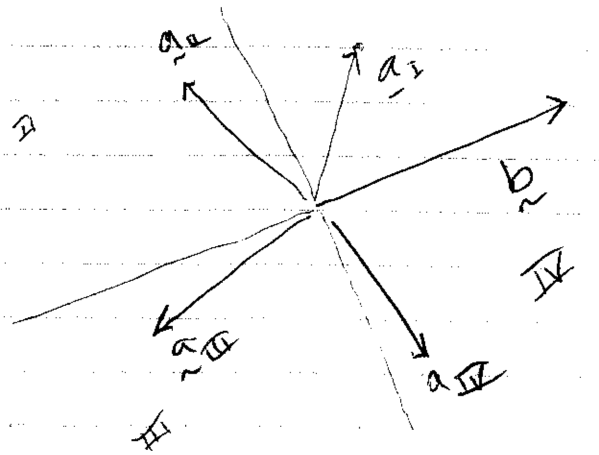
E.g. $(a_1, a_2, 0)$ in space, how many $\underset{\sim}{b} = (b_1, b_2, b_3)$ are orthogonal?

Same as before, but also $\underset{\sim}{b} = (0, 0, b_3)$

* note * if $\underset{\sim}{b_i} \cdot \underset{\sim}{a} = 0$, then

$$\left(\sum_{i=1}^{n} c_i \underset{\sim}{b_i}\right) \cdot \underset{\sim}{a} = 0$$

Sign of the dot product



I. $a_I \cdot b \geq 0$

II. $a_{II} \cdot b \leq 0$

III. $a_{III} \cdot b \leq 0$

IV. $a_{IV} \cdot b \geq 0$

Direction Cosines of $a = (a_1, a_2, a_3)$

α - angle between a and the x-axis
β - y-axis
γ - z-axis

$$\cos(\alpha) = \frac{a \cdot i}{|a||i|} = \frac{a_1}{|a|}$$

$$\cos(\beta) = \frac{a \cdot j}{|a||j|} = \frac{a_2}{|a|}$$

$$\cos(\gamma) = \frac{a \cdot k}{|a||k|} = \frac{a_3}{|a|}$$

so $\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = \frac{|a|^2}{|a|^2} = 1$

Projection

rb

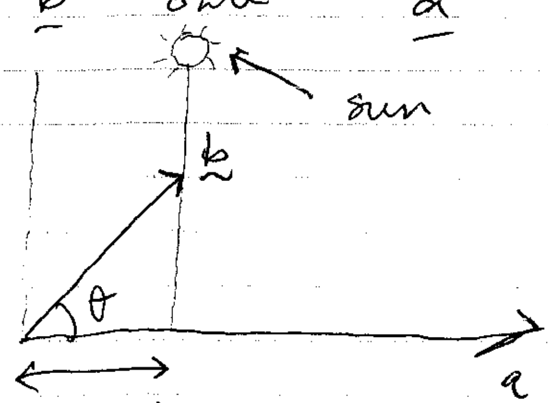
- how much is one vector like another?
- what is the "closest" vector to a vector $\underset{\sim}{b}$ in the direction $\underset{\sim}{a}$?
- general tools for analyzing vector data

lb

Scalar Projection

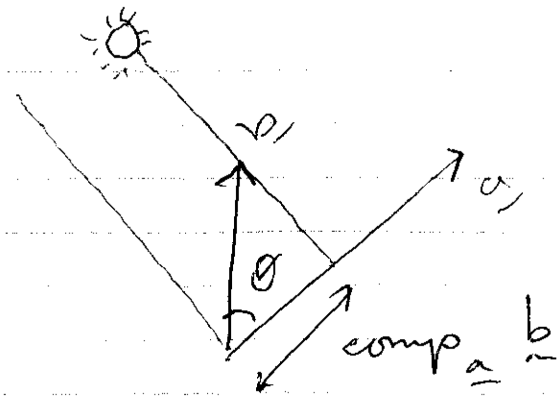
- component of $\underset{\sim}{b}$ along $\underset{\sim}{a}$
- size of shadow cast by $\underset{\sim}{b}$ onto $\underset{\sim}{a}$

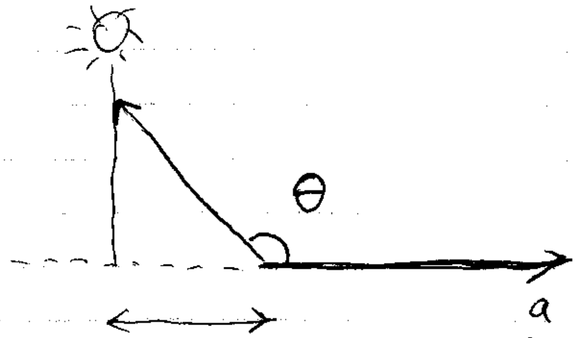
llb



$\text{comp}_{\underset{\sim}{a}}\,\underset{\sim}{b}$

$$= |\underset{\sim}{b}| \cos(\theta)$$





$$\text{comp}_{\underset{\sim}{a}}\, \underset{\sim}{b} = |\underset{\sim}{b}| \cos(\theta) < 0$$

Using dot products

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$$

$$\therefore \quad \text{comp}_{\underset{\sim}{a}}\, \underset{\sim}{b} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|}$$

rb

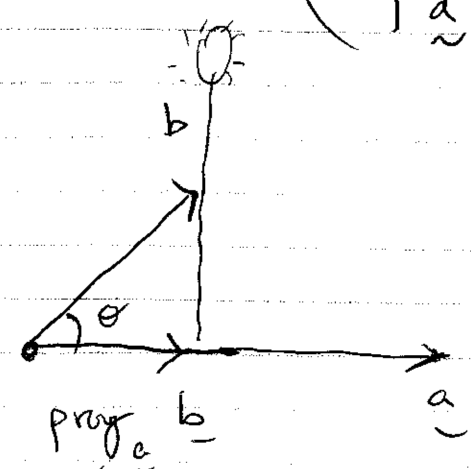
Vector Projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$

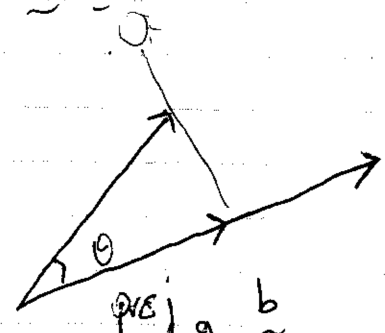
– closest vector in direction $\underset{\sim}{a}/|\underset{\sim}{a}|$ to vector $\underset{\sim}{b}$

$$\text{proj}_{\underset{\sim}{a}} \underset{\sim}{b} = \text{comp}_{\underset{\sim}{a}} \underset{\sim}{b} \; \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

$$= \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2} \right) \underset{\sim}{a}$$

ls





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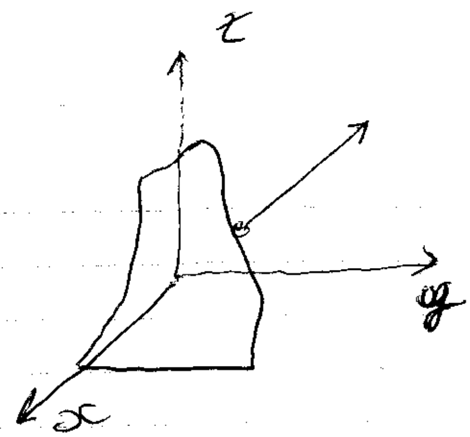
$\boxed{b}$



large rock on ice

$$\underset{\sim}{f} = (5, 9, 3)$$

$$\underset{\sim}{f} \cdot \underset{\sim}{k} = 3$$

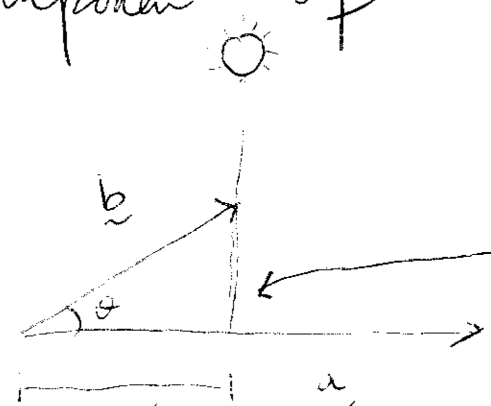
$$\underset{\sim}{f} \cdot \underset{\sim}{j} = 9$$

$$\underset{\sim}{f} \cdot \underset{\sim}{i} = 5$$

$$\underset{\sim}{f} - \text{proj}_{\underset{\sim}{k}} f = (5, 9, 0)$$

Component + Projection   10/3/08   -1

I) - component of $\mathbf{b}$ on $\mathbf{a}$



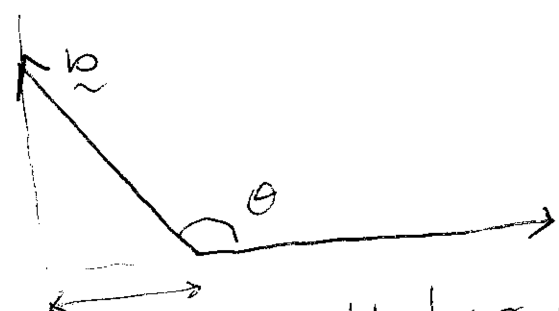
- $\text{comp}_{\mathbf{a}}\,\mathbf{b}$ = shadow cast = comp of $\mathbf{b}$ on $\mathbf{a}$

$\text{comp}_{\mathbf{a}}\,\mathbf{b}$

$$= |\mathbf{b}|\cos(\theta)$$

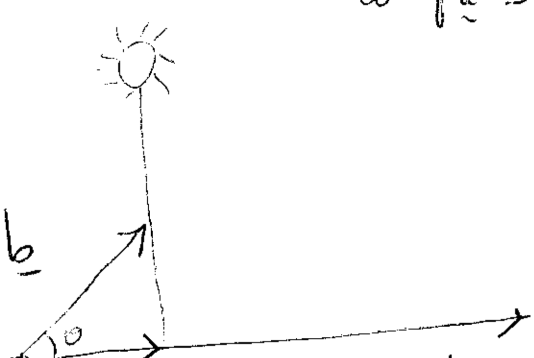
$$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Ib - note



$\text{comp}_{\mathbf{a}}\,\mathbf{b} = |\mathbf{b}|\cos\theta$

IIb Projection



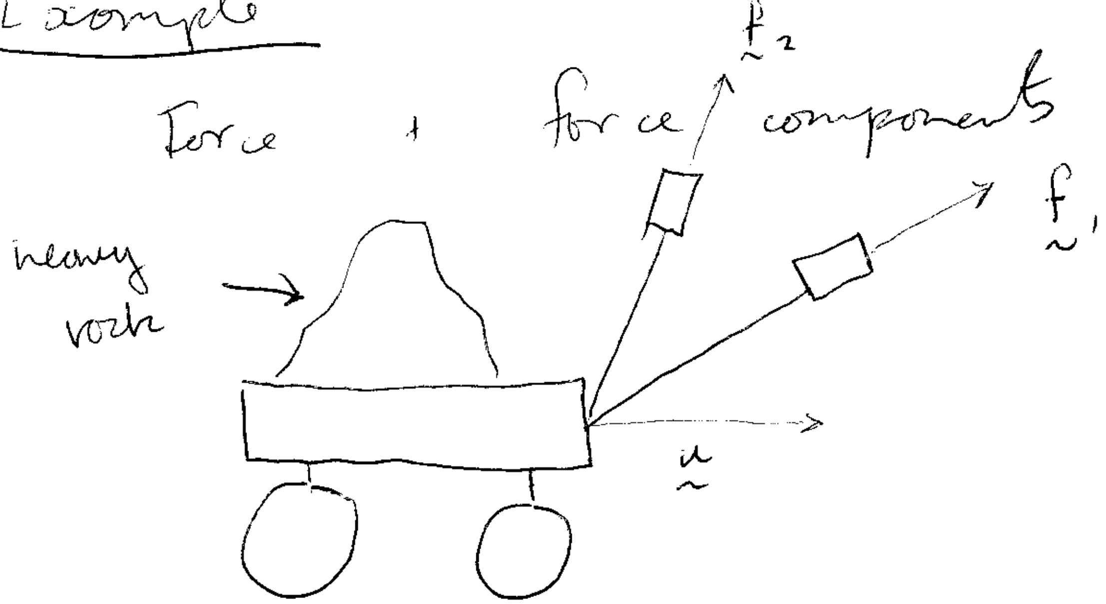
$|\text{proj}_{\mathbf{a}}\,\mathbf{b}| = \text{comp}_{\mathbf{a}}\,\mathbf{b}$

$\text{proj}_{\mathbf{a}}\,\mathbf{b}$ = projection of $\mathbf{b}$ on $\mathbf{a}$

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = (\text{comp}_{\mathbf{a}}\,\mathbf{b})\cdot \mathbf{a}/|\mathbf{a}| = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

Example

a) Force + force components



- effective force on wagon is

$$\left(\frac{\underset{\sim}{f} \cdot \underset{\sim}{u}}{|\underset{\sim}{u}|} \right) \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$$

$\boxed{b}$ for any $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ w/ $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & i = j \end{cases}$

every vector $\underset{\sim}{v}$ can be written

as
$$\underset{\sim}{v} = (\text{comp}_{\underset{\sim}{e}_1} \underset{\sim}{v})\, \underset{\sim}{e}_1 + (\text{comp}_{\underset{\sim}{e}_2} \underset{\sim}{v})\, \underset{\sim}{e}_2 + (\text{comp}_{\underset{\sim}{e}_3} \underset{\sim}{v})\, \underset{\sim}{e}_3$$

The Cross Product 10/3/08 ①

rb - $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{c}$ ← vector

- algebraic definition, geometric interpretation

- uses: computing the torque ($\underset{\sim}{\tau}$) on an object from a force ($\underset{\sim}{f}$) on the object

lb Definition: If $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$, then the cross product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Determinants

rb - 2nd order

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

e.g.

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2 \cdot 4 - 1 \cdot (-6) = 14$$

$$\begin{vmatrix} 3 & 2 \\ 0 & 5 \end{vmatrix} = 15$$

lb 3rd order

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$\boxed{\text{rb}}$ $\underset{\sim}{a} \times \underset{\sim}{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\ -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

so $$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underset{\sim}{k}$$

$\boxed{\text{lb}}$ alternatively

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underset{\sim}{k}$$

11b

$$a\,\tilde{i} \times b\,\tilde{j} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ a & 0 & 0 \\ 0 & b & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \tilde{k} = ab\,\tilde{k}$$

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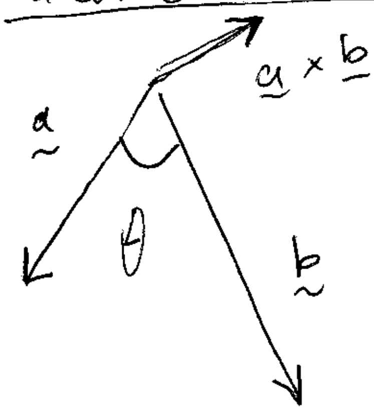
Geometric Interpretation



\- right-hand rule

(i) allign fingers w/ $\tilde{a}$

(ii) curl fingers towards $\tilde{b}$ (take the shorter route)

* in the process, your thumb will be pointing in the direction of $\tilde{a} \times \tilde{b}$

$\boxed{\text{nb}}$ right-hand rule: why?

- can show $(\underline{a} \times \underline{b}) \cdot \underline{a} = 0$

+ $(\underline{a} \times \underline{b}) \cdot \underline{b} = 0$

* note: $c\,\underline{a} + d\,\underline{b}$ are in the plane determined by $\underline{a}$ and $\underline{b}$

$\boxed{\text{nb}}$ $(\underline{a} \times \underline{b}) \cdot (c\,\underline{a} + d\,\underline{b}) =$

$$a\,(\underline{a} \times \underline{b}) \cdot \underline{a} + d\,(\underline{a} \times \underline{b}) \cdot \underline{b} = 0$$

i.e. the vector $\underline{a} \times \underline{b}$ is orthogonal to all vectors in the plane determined by $\underline{a}$ and $\underline{b}$

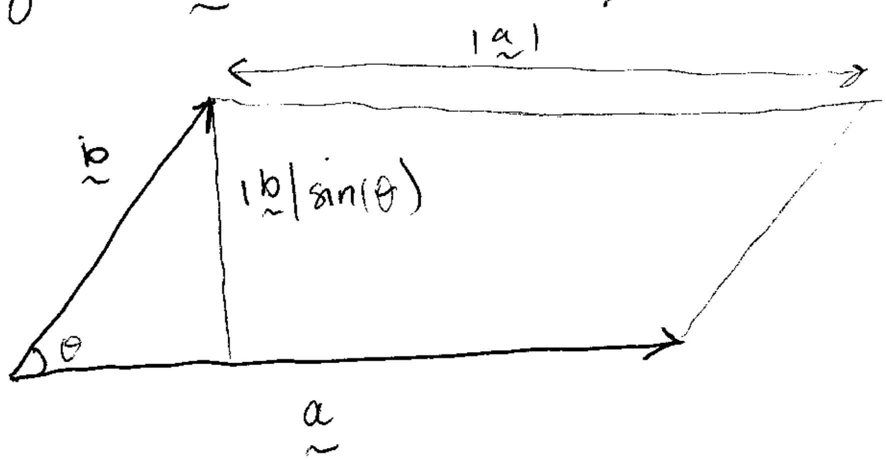
rb **Theorem**: If θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin(\theta)$

* note * $0 \le \theta \le \pi$

* note * $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{0}$ if $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

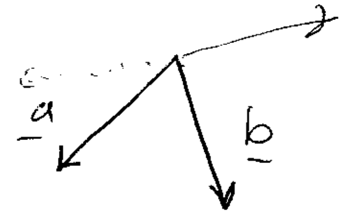
Mb Geometry

The length of $\underset{\sim}{a} \times \underset{\sim}{b}$ is the area of the parallelogram determined by $\underset{\sim}{a}$ and $\underset{\sim}{b}$



NB Properties of the crossproduct (7)

1. $\underset{\sim}{a} \times \underset{\sim}{b} = - \underset{\sim}{b} \times \underset{\sim}{a}$



2. $(c\underset{\sim}{a}) \times \underset{\sim}{b} = c(\underset{\sim}{a} \times \underset{\sim}{b})$
$= \underset{\sim}{a} \times (c\underset{\sim}{b})$

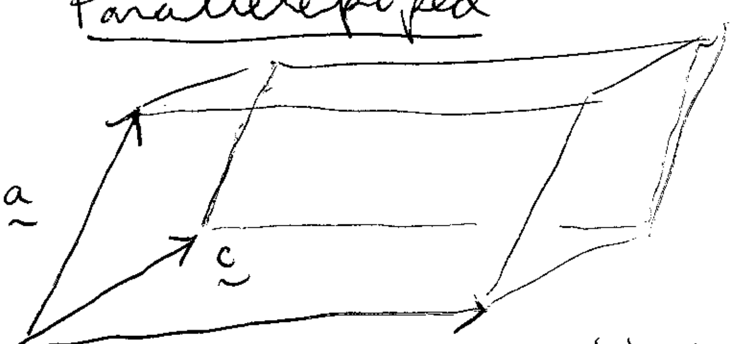
3. $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$

4 $(\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{c}$

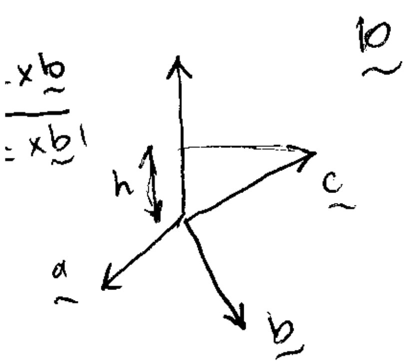
5. $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}$

6 $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

NB Parallelepiped



$V = |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$



$(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

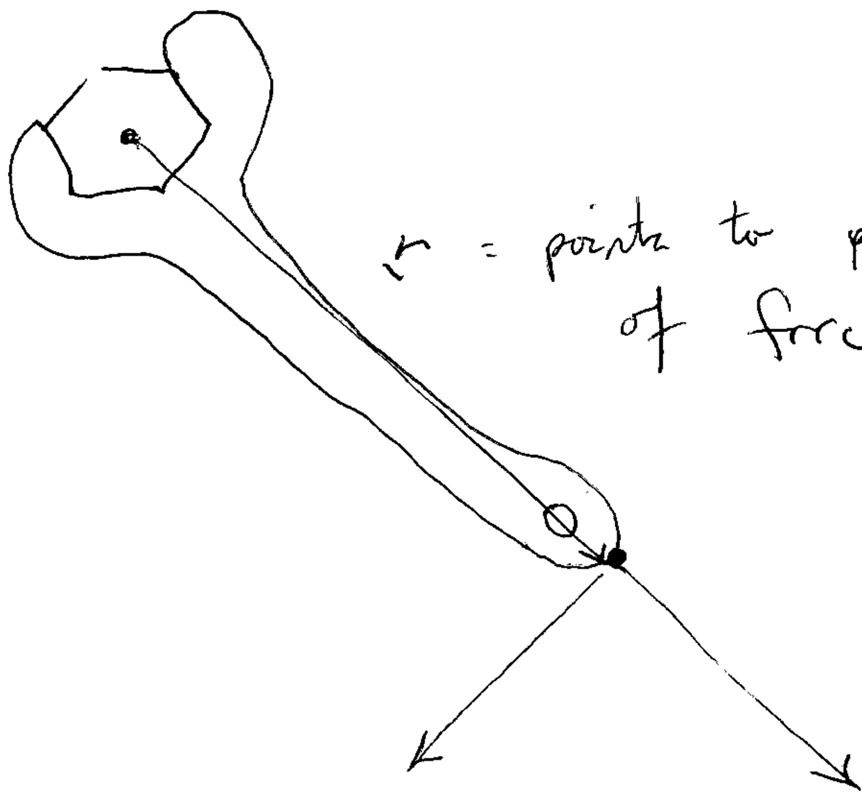
$h = \left(\dfrac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a} \times \underset{\sim}{b}|} \right)$

$= h \cdot A$

$\boxed{rb}$

Torque

$$\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{f}$$



$\underset{\sim}{r}$ = points to point of application of force