Math 32 A: Midterm 2, November 14, 2008

- Name:
- ID Number:
- Section:

problem	score	possible
1.		10
2.		10
3.		10
4.		10
5.		10
Total		50

1. Shade the region of the xy plane representing the domain of the following functions?

   a. $f(x,y) = \frac{1}{\ln(x^2+y^2-1)}$

   b. $f(x,y) = \frac{\ln(xy)}{2xy-1}$

   Solution:

   a. $f(x,y)$ is defined wherever the denominator is defined and nonzero. The denominator $\ln(x^2+y^2-1)$ is defined for $x^2+y^2-1>0$ and is nonzero for $x^2+y^2 \neq 2$. Thus, the domain of the function is the region outside the circle of radius 1 (centered at the origin), excluding the circle of radius $\sqrt{2}$, as shown in the figure below.

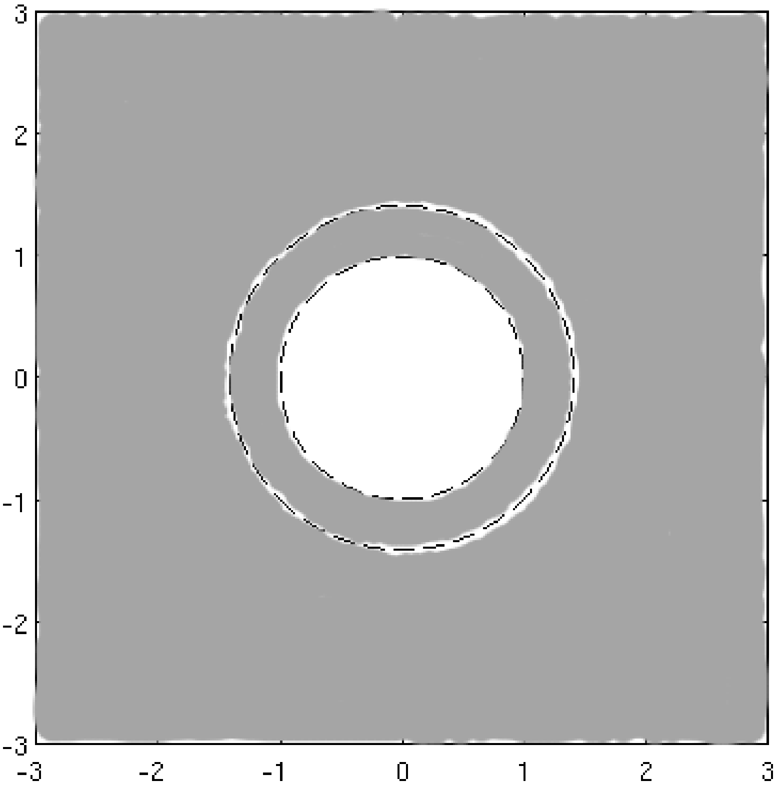

Figure 1: Domain of $f(x,y) = \frac{1}{\ln(x^2+y^2-1)}$

b. $f(x, y)$ is defined in the region where the numerator $\ln(xy)$ is defined and the denominator $2xy - 1$ is nonzero. This is the region in $\mathbb{R}^2$ satisfying $xy > 0$ and $y \neq \frac{1}{2x}$, as shown in the figure below.

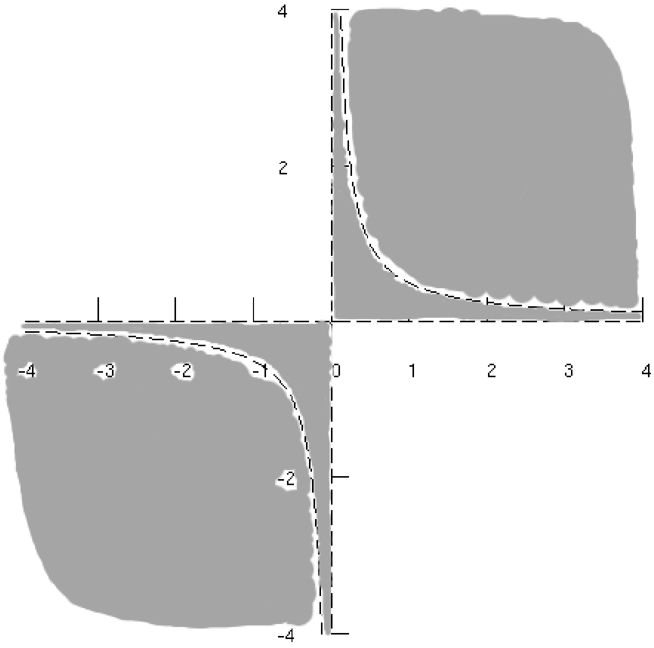

Figure 2: Domain of $f(x, y) = \frac{\ln(xy)}{2xy-1}$

2. Find the limit of the following functions if they exist

   a. $\lim\limits_{(x,y)\to(0,0)} \frac{x}{e^{xy}+y}$

   b. $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2}$

   Solution:

   a. The first thing to notice here is that the numerator and denominator are both continuous at $(x, y) = (0, 0)$, and the denominator is nonzero there. So

$$\lim_{(x,y)\to(0,0)} \frac{x}{e^{xy}+y} = \frac{\lim\limits_{(x,y)\to(0,0)} x}{\lim\limits_{(x,y)\to(0,0)} e^{xy}+y} = \frac{0}{1} = 0.$$

   b. Suppose the limit exists and equals L. Then the limit along any line passing through the origin must equal L, which implies that we should get the same limit regardless of which line we take. We will see that this is not the case.

   First we consider the line $y = 0$. Then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2} = L \implies L = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

   Now we consider the line $y = x$, and we see that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2} = L \implies L = \lim_{x\to 0} \frac{x^2}{4x^2} = \frac{1}{4}.$$

   So if the limit exists, then $L = 0 = \frac{1}{4}$, an obvious absurdity. Hence we conclude that the limit does not exist.

3. a. Let $f(x) = -4x + 1$. Show that for every $\epsilon > 0$ there is a δ such that $|f(x) + 3| < \epsilon$, if $|x - 1| < \delta$.

b. Let $f(x) = \begin{cases} 3x + 2 & \text{if } x \leq 2 \\ 7 & \text{if } x > 2 \end{cases}$.

Show that if $\epsilon = \frac{1}{2}$, there is no $\delta > 0$ such that $|f(x) - 8| < \epsilon$, if $|x - 2| < \delta$.

Solution:

a. Let $\delta = \frac{\epsilon}{4}$. Then we have,

$$|f(x) + 3| = |-4x + 1 + 3| = |-4x + 4| = |-4(x - 1)|$$

$$= |-4||x - 1| = 4|x - 1| < 4\delta = \epsilon.$$

b. Suppose a δ did exist so that $|f(x) - 8| < \frac{1}{2}$ if $|x - 2| < \delta$. For such a δ, let $x = 2 + \frac{\delta}{2}$. Then $|x - 2| = \frac{\delta}{2} < \delta$, giving $|f(x) - 8| < \frac{1}{2}$. On the other hand, since $2 + \frac{\delta}{2} > 2$,

$$|f(x) - 8| = |7 - 8| = 1.$$

So, if a δ existed, we have simultaneously $|f(x) - 8| < \frac{1}{2}$ and $|f(x) - 8| = 1$ for $x = 2 + \frac{\delta}{2}$. This is impossible, so no δ can exist.

4. Let $z = f(x,y)$ where

$$xyz + x + y + z^5 + 3 = 0$$

a. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

b. Give an equation for the tangent plane to the surface define by $z = f(x,y)$ at the point $(-\frac{5}{2}, 1, 1)$.

Solution:

a. We use implicit differentiation to solve for each derivative. First, differentiating both sides with respect to x we obtain

$$yz + yx\frac{\partial z}{\partial x} + 1 + 5z^4\frac{\partial z}{\partial x} = 0$$

where we have applied the product rule to the product of functions $xyz = (xy)(z)$ and treat y as a constant. Combining the $\frac{\partial z}{\partial x}$ terms and solving for it we obtain

$$\frac{\partial z}{\partial x}(yx + 5z^4) = -(1 + yz) \implies \frac{\partial z}{\partial x} = \frac{-(1+yz)}{yx + 5z^4}.$$

Proceeding analogously but now differentiating with respect to y and keeping x constant (and with the same product rule trick as above) we obtain

$$xz + yx\frac{\partial z}{\partial y} + 1 + 5z^4\frac{\partial z}{\partial y} = 0.$$

Solving for $\frac{\partial z}{\partial y}$ we obtain

$$\frac{\partial z}{\partial y} = \frac{-(1+xz)}{yx + 5z^4}.$$

Notice they are symmetric in x and y as they should be since our original relation is.

b. From the text (p. 928) we have the formula for the tangent plane to a function $z = f(x,y)$ at the point (x_0, y_0, z_0) is given by the formula

$$z - z_0 = f_x(x_0, y)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Although in this formula the partials are functions of two variables and our partials are functions of three, the z value is completely determined, in fact given here, by the x and y. Therefore, plugging in our point $(-5/2, 1, 1)$ above we have the tangent plane is given by

$$z - 1 = \frac{-(1+1)}{-\frac{5}{2} + 5}(x + \frac{5}{2}) + \frac{-(1 - \frac{5}{2})}{-\frac{5}{2} + 5}(y - 1).$$

5. Let $\mathbf{r_1}(t) = (\frac{t^2}{2}, t-1, t^3-3)$ and $\mathbf{r_2}(s) = (3s^2+s+2, 1-s^2, s+5)$ be two curves in a surface S_1 which intersect at the point $P_1 = (2,1,5)$. Let

$$\mathbf{r_3}(u) = (2u^2-4u, \frac{2-u}{2}, u^3+u^2-3u)$$

be a third curve which is ***normal*** to another surface S_2 at the point $P_2 = (-2, 1/2, -1)$.

a. Find an equation for the tangent plane to S_1 at P_1 and an equation for the tangent plane to S_2 at P_2.

b. Find the angle between the tangent planes in part (a).

Solution:

a. First, we need to figure out at what time all of the curves pass through the given points. Notice, each given curve has one coordinate that is just a linear function of the parametrizing variable. So, we should use that coordinate and the corresponding coordinate for the given point to solve for the specific time variable. Since $\mathbf{r_1}$ and $\mathbf{r_2}$ both intersect $(2,1,5)$ we need to solve

$$\begin{aligned} t-1=1 &\implies t=2 \\ s+5=5 &\implies s=0. \end{aligned}$$

You can check that these values of t and s do in fact make the curves pass through the given point, i.e. the remaining coordinate values of the curves are correct. Doing the same thing for $\mathbf{r_3}$ we solve

$$\frac{2-u}{2} = \frac{1}{2} \implies u=1.$$

Now, to get the equations of the two tangent planes we need to only find normal vectors to each because we already have points in each; those are the given points. For the first surface S_1, since $\mathbf{r_1}$ and $\mathbf{r_2}$ are in S_1 and intersect at P_1 their *tangent* vectors at that point are both in the tangent plane. So if we cross their tangent vectors it will be the normal we need. So,

$$\begin{aligned} \mathbf{r_1'}(t) = \langle t, 1, 3t^2\rangle &\implies \mathbf{r_1'}(2) = \langle 2,1,12\rangle \\ \mathbf{r_2'}(s) = \langle 6s+1, -2s, 1\rangle &\implies \mathbf{r_2'}(0) = \langle 1,0,1\rangle. \end{aligned}$$

These are two vectors in the first tangent plane, so we cross them to get a normal. So,

$$\mathbf{N_1} = \mathbf{r_1'}(2) \times \mathbf{r_2'}(0) = \mathbf{i} + 10\mathbf{j} - \mathbf{k} = \langle 1, 10, -1\rangle$$

So, the equation of the first tangent plane is

$$(x-2) + 10(y-1) - (z-5) = 0.$$

To get the normal to the second tangent plane we don't need to cross anything because by hypothesis, $\mathbf{r_3}$ is already normal to S_2 at the given point. What this means is precisely that it is normal to the tangent plane to S_2 at that point; the tangent plane is locally the orientation of the surface, the way it locally sits in space around that point. Thus, to say that $\mathbf{r_3}$ is normal to S_2 at P_2 really means that it is normal to the tangent plane to S_2 at P_2. Thus,

$$\mathbf{N_2} = \mathbf{r}_\mathbf{3}'(1) = \left\langle 4u-4, -\frac{1}{2}, 3u^2+2u-3 \right\rangle\Big|_{u=1} = \left\langle 0, -\frac{1}{2}, 2 \right\rangle.$$

So, the equation of the second tangent plane is

$$-\frac{1}{2}\left(y-\frac{1}{2}\right)+2(z+1)=0.$$

b. The angle between the tangent planes is the same as the angle between their normal vectors (see Figure 9 on page 835 of the text). So, we use the dot product of $\mathbf{N_1}$ and $\mathbf{N_2}$ above to solve for the angle between them.

$$\mathbf{N_1}\cdot\mathbf{N_2} = |\mathbf{N_1}||\mathbf{N_2}|\cos(\theta).$$

So,

$$\langle 1, 10, -1\rangle \cdot \left\langle 0, -\frac{1}{2}, 2\right\rangle = -5-2 = -7 = \sqrt{1+10^2+1}\sqrt{\left(\frac{1}{2}\right)^2+2^2}\ \cos(\theta).$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{102}\sqrt{(1/4)+4}}\right),$$

and no, you don't have to simplify this.