

**Math 32 A: Midterm 1, Oct. 24, 2008**

- Name:
- ID Number:
- Section:

problem	score	possible
1.		10
2.		10
3.		10
4.		10
5.		10
6.		10
Total		60

1. Determine whether the following points are coplanar:

a.  $\mathbf{P} = (8, 14, 12)$ ,  $\mathbf{Q} = (11, 19, 21)$ ,  $\mathbf{R} = (6, 11, 2)$ ,  $\mathbf{S} = (5, 9, 3)$

b.  $\mathbf{P} = (2, 0, -1)$ ,  $\mathbf{Q} = (4, 1, 0)$ ,  $\mathbf{R} = (3, -1, 1)$ ,  $\mathbf{S} = (2, -2, 2)$

**Solution:** The points are coplanar if and only if the volume of the parallelepiped determined by the vectors  $\overrightarrow{\mathbf{PQ}}$ ,  $\overrightarrow{\mathbf{PR}}$ ,  $\overrightarrow{\mathbf{PS}}$  is zero. That is, if and only if  $\overrightarrow{\mathbf{PS}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PQ}}) = 0$ .

a.  $\overrightarrow{\mathbf{PQ}} = \langle 3, 5, 9 \rangle$ ,  $\overrightarrow{\mathbf{PR}} = \langle -2, -3, -10 \rangle$ ,  $\overrightarrow{\mathbf{PS}} = \langle -3, -5, -9 \rangle$ . So,  $\overrightarrow{\mathbf{PS}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PQ}}) = 0$  and the points are coplanar.

b.  $\overrightarrow{\mathbf{PQ}} = \langle 2, 1, 1 \rangle$ ,  $\overrightarrow{\mathbf{PR}} = \langle -1, -2, 1 \rangle$ ,  $\overrightarrow{\mathbf{PS}} = \langle 0, -2, 3 \rangle$ . So,  $\overrightarrow{\mathbf{PS}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PQ}}) = 3$  and the points are not coplanar.

2. The line  $L$  has parametric equations:

$$\begin{aligned}x(t) &= 3 + 2t \\y(t) &= 4 - t \\z(t) &= 5 + 6t.\end{aligned}$$

Find the closest point on the line  $L$  to the point  $\mathbf{P} = (4, 1, 7)$ .

**Solution:** What we want to do is write an equation for the vector from our fixed point  $(4, 1, 7)$  to an *arbitrary* point on the line. The shortest distance will be when this vector is perpendicular to the direction of the line, because moving in a direction perpendicular to the line gets you there in the shortest distance; you don't waste any distance moving in directions parallel to the line. That is, the perpendicular distance is the shortest distance. So, what is this vector? Well, an arbitrary point on the line has coordinates  $(x(t), y(t), z(t)) = (3 + 2t, 4 - t, 5 + 6t)$  hence

$$\mathbf{w} = \langle (3 + 2t, 4 - t, 5 + 6t) - (4, 1, 7) \rangle = \langle 2t - 1, 3 - t, 6t - 2 \rangle$$

is the vector that points from  $(4, 1, 7)$  to an arbitrary point on the line, determined by each possible value of  $t$ . Using the parametric equations for  $L$  we can read off its direction vector, which has coordinates the coefficients of  $t$ . So  $\mathbf{v} = \langle 2, -1, 6 \rangle$  lies in the direction of  $L$  and we want  $\mathbf{w}$  to be perpendicular to it. So, we need to solve

$$0 = \mathbf{v} \cdot \mathbf{w} = \langle 2, -1, 6 \rangle \cdot \langle 2t - 1, 3 - t, 6t - 2 \rangle = 4t - 2 + t - 3 + 36t - 12 = 41t - 17$$

hence the shortest distance occurs when  $t = 17/41$ . Plugging this into the parametric equations we see that this corresponds to the point  $(3 + 2 \cdot 17/41, 4 - 17/41, 5 + 6 \cdot 17/41)$ . A less geometric approach would be to minimize  $|\mathbf{w}|$  directly by setting its derivative equal to 0. Note, we would differentiate  $|\mathbf{w}|$  and not  $\mathbf{w}$  itself.

3. Line  $L_1$  has parametric equations:

$$\begin{aligned}x(t) &= 2 + 3t \\y(t) &= 1 - 5t \\z(t) &= -3 + 2t.\end{aligned}$$

Line  $L_2$  has parametric equations:

$$\begin{aligned}x(s) &= 3 + 2s \\y(s) &= 1 + 6s \\z(s) &= 5 + 7s.\end{aligned}$$

Find the distance between the two lines  $L_1$  and  $L_2$ .

**Solution:** First we find a unit vector that is perpendicular to both lines. For that we take the cross product of two vectors, one in the direction of line  $L_1$  and the other in the direction of line  $L_2$ .

Vector in the direction of  $L_1$  :

$$\vec{v}_1 = \langle 3, -5, 2 \rangle.$$

Vector in the direction of  $L_2$  :

$$\vec{v}_2 = \langle 2, 6, 7 \rangle.$$

Thus a normal to both lines will be given by

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -47, -17, 28 \rangle.$$

The unit normal is then given by

$$\frac{\vec{n}}{|\vec{n}|} = \frac{\langle -47, -17, 28 \rangle}{\sqrt{47^2 + 17^2 + 28^2}}.$$

To obtain the distance between the lines, we take the dot product of this unit normal with any vector  $\vec{r}_2 - \vec{r}_1$ , where  $\vec{r}_2$  is the position vector of an arbitrary point on  $L_2$  and  $\vec{r}_1$  is the position vector of an arbitrary point on  $L_1$ . We can take  $\vec{r}_2 = \langle 3, 1, 5 \rangle$  and  $\vec{r}_1 = \langle 2, 1, -3 \rangle$ , and so the desired distance is

$$\langle 1, 0, 8 \rangle \cdot \frac{\langle -47, -17, 28 \rangle}{\sqrt{47^2 + 17^2 + 28^2}} = \frac{177}{\sqrt{47^2 + 17^2 + 28^2}}.$$

4. Show that the paths  $\mathbf{G}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$  and  $\mathbf{H}(t) = t^5\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$  intersect at time  $t=1$ . What is the angle between the paths at the point of intersection?

**Solution:** We have  $\mathbf{G}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{H}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ . Since  $\mathbf{G}(1) = \mathbf{H}(1)$  the paths intersect at time  $t=1$ .

The angle between the curves at the point of intersection is the angle  $\theta$  between their tangent vectors at time  $t = 1$ . Since  $\mathbf{G}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$  and  $\mathbf{H}'(t) = 5t^4\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ , we have  $\mathbf{G}'(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$  and  $\mathbf{H}'(1) = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ . So,

$$\theta = \cos^{-1} \left( \frac{\mathbf{G}'(1) \cdot \mathbf{H}'(1)}{|\mathbf{G}'(1)||\mathbf{H}'(1)|} \right) = \cos^{-1} \left( \frac{12}{\sqrt{14}\sqrt{30}} \right).$$

5. Consider the cycloid path of a particle on a rolling (from right to left) wheel of radius  $r$  given by  $\mathbf{r}(t) = (-2\pi r t, r) + (r \cos(2\pi t), r \sin(2\pi t))$ .

a. What is the maximum speed of the particle? At what time  $t$  ( $0 \leq t \leq 1$ ) does this happen?

b. What is the minimum speed of the particle? At what time  $t$  ( $0 \leq t \leq 1$ ) does this happen?

**Solution:**

a. To obtain an expression for the speed, we first obtain the velocity vector by differentiating  $\mathbf{r}(t)$ . The velocity vector is then:

$$\mathbf{r}'(t) = -2\pi r \langle 1 + \sin(2\pi t), -\cos(2\pi t) \rangle$$

Taking the magnitude:

$$\begin{aligned} |\mathbf{r}'(t)| &= 2\pi r |\langle 1 + \sin(2\pi t), -\cos(2\pi t) \rangle| \\ &= 2\pi r \sqrt{(1 + \sin 2\pi t)^2 + \cos^2(2\pi t)} = 2\pi r \sqrt{2 + 2 \sin 2\pi t}. \end{aligned}$$

It is clear that this expression attains a maximum when  $2 \sin 2\pi t$  is maximized. This happens at  $t = 1/4$ , corresponding to a maximum speed of  $4\pi r$ .

b. Looking at the expression above, we can see that it attains a minimum when  $2 \sin 2\pi t$  is minimized. This happens at  $t = 3/4$ , corresponding to a minimum speed of 0.

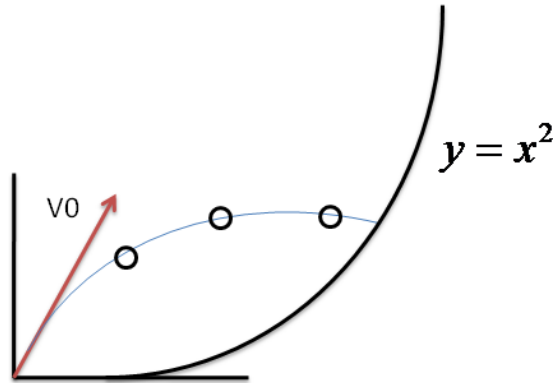


Figure 1: Problem 6 illustration.

6. Consider a projectile shot with initial velocity  $\mathbf{v}_0$  above a slope rising from the horizontal with profile  $y(x) = x^2$  (see Figure 1).
- How long does it take the projectile to hit the slope? Write your answer in terms of gravitational constant  $g$  and  $\mathbf{v}_0 = (v_{0x}, v_{0y})$ .
  - What is the normal to the slope ( $\mathbf{n}$ ) at the point of contact? What is the normal component of the projectile velocity when it makes contact (i.e. what is  $\text{comp}_{\mathbf{n}} \mathbf{v}$ )?

**Solution:**

a. To find the time of intersection, we parametrize both the path of the projectile and the slope, and then intersect these two parametrizations. From the text book (p. 877) we have that the parametrization for the projectile is given by

$$\begin{aligned} x(t) &= v_{0x}t \\ y(t) &= v_{0y}t - \frac{1}{2}gt^2. \end{aligned}$$

Now, we parametrize the slope, but we *must* use a different variable of parametrization. Thus, the slope is parametrized by

$$\begin{aligned} x(s) &= s \\ y(s) &= s^2. \end{aligned}$$

To find intersection we set

$$\begin{aligned} v_{0x}t &= s \\ v_{0y}t - \frac{1}{2}gt^2 &= s^2. \end{aligned}$$

Substituting the first equation into the second we reduce to  $v_{0y}t - \frac{1}{2}gt^2 = (v_{0x}t)^2 = v_{0x}^2t^2$ . To solve for  $t$  we have

$$v_{0x}^2t^2 + \frac{1}{2}gt^2 - v_{0y}t = 0 \implies t(v_{0x}^2t + \frac{1}{2}gt - v_{0y}) = 0.$$

Clearly  $t = 0$  is a solution, but this isn't the one you are after (i.e, the projectile is intersecting the slope before it takes off at the origin). So you, have the other solution

$$v_{0x}^2t + \frac{1}{2}gt - v_{0y} = 0 \implies t(v_{0x}^2 + \frac{1}{2}g) = v_{0y} \implies t = \frac{v_{0y}}{v_{0x}^2 + \frac{1}{2}g}.$$

b. To do this, the fastest way is to first find the tangent vector to the slope at the point of intersection and then find the perpendicular to that. We parametrized our slope above with  $\mathbf{r}(s) = \langle s, s^2 \rangle$  so our tangent vector to the slope is given by  $\mathbf{T}(s) = \langle 1, 2s \rangle$ . From part (a) we have our relation  $s = v_{0x}t$  and we know the  $t$  value at the point of intersection; that was precisely the whole point of part (a)! So, we plug in our answer from part (a) to get the actual tangent vector we want:

$$\mathbf{T} = \langle 1, 2s \rangle = \langle 1, 2v_{0x}t \rangle = \left\langle 1, \frac{2v_{0x}v_{0y}}{v_{0x}^2 + \frac{1}{2}g} \right\rangle$$

Now, we have our tangent vector at the point of intersection and we need to find a vector perpendicular to it. If we let the  $x$  component of our  $\mathbf{N}$  be  $-1$  then we want the product of its  $y$  component and the  $y$  component of  $\mathbf{T}$  to be 1 so that the dot product is 0. Thus, we just take the reciprical of the  $y$  component of  $\mathbf{T}$  and make that the  $y$  component of  $\mathbf{N}$ . So, letting

$$\mathbf{N} = \left\langle -1, \frac{v_{0x}^2 + \frac{1}{2}g}{2v_{0x}v_{0y}} \right\rangle$$

we can see that  $\mathbf{T} \cdot \mathbf{N} = 0$ . To finish we want to find  $\text{comp}_{\mathbf{N}}\mathbf{v}$ . We have that the curve of the projectile is given by  $\mathbf{r}(t) = \langle v_{0x}t, v_{0y}t - 1/2gt^2 \rangle$  therefore  $\mathbf{v}(t) = \langle v_{0x}, v_{0y} - gt \rangle$ . Therefore

$$\text{comp}_{\mathbf{N}}\mathbf{v} = \frac{\mathbf{N} \cdot \mathbf{v}}{|\mathbf{N}|} = \frac{\left\langle -1, \frac{v_{0x}^2 + \frac{1}{2}g}{2v_{0x}v_{0y}} \right\rangle \cdot \langle v_{0x}, v_{0y} - gt \rangle}{\sqrt{1 + \left( \frac{v_{0x}^2 + \frac{1}{2}g}{2v_{0x}v_{0y}} \right)^2}} = \frac{-v_{0x} + (v_{0y} - gt)\left(\frac{v_{0x}^2 + \frac{1}{2}g}{2v_{0x}v_{0y}}\right)}{\sqrt{1 + \left( \frac{v_{0x}^2 + \frac{1}{2}g}{2v_{0x}v_{0y}} \right)^2}},$$

and no, you don't have to simplify this.