Inverse deformation gradient

1 Deformation gradient

Let $\phi(\cdot, t) : \Omega^0 \to \Omega^t$ be the flow map of our material. The deformation gradient $F(\cdot, t) : \Omega^0 \to \mathbb{R}^{2 \times 2}$ is defined as

$$F(X, t) = \frac{\partial \phi}{\partial X}(X, t), \text{ or } F_{ij}(X, t) = \frac{\partial \phi_i}{\partial X_j}(X, t).$$

2 Inverse material mapping

If $\phi(\cdot, t) : \Omega^0 \to \Omega^t$ is the flow map of our material, let $\phi^{-1}(\cdot, t) : \Omega^t \to \Omega^0$ be the inverse flow map. Then, by definition

$$X = \phi^{-1}(\phi(X, t), t), \forall X \in \Omega^0, \forall t \geq 0$$

and

$$x = \phi(\phi^{-1}(x, t), t), \forall x \in \Omega^t, \forall t \geq 0.$$

3 Inverse deformation gradient

The inverse of the deformation gradient $F^{-1}(\cdot, t) : \Omega^0 \to \mathbb{R}^{2 \times 2}$ is the mapping defined as

$$F^{-1}(X, t) = (F(X, t))^{-1}$$

In other words, at any point $X \in \Omega^0$ and at any time $t \geq 0$, the function $F^{-1}(X, t)$ is defined to be the inverse of the deformation gradient at that point and time.

4 Deformation gradient of inverse mapping

We can also define $\frac{\partial \phi^{-1}}{\partial x}(\cdot, t) : \Omega^t \to \mathbb{R}^{2 \times 2}$ as a function over $\Omega^t$. It is just the Jacobian of the inverse mapping. I sometimes use the notation

$$\frac{\partial \phi^{-1}}{\partial x} = \frac{\partial X}{\partial x}, \text{ or } \frac{\partial \phi^{-1}_i}{\partial x_j}.$$

5 Inverse of the deformation gradient is the pull back of the inverse deformation gradient

I know that sounds confusing, but it is an important relation. That is, the inverse of the deformation gradient $F^{-1}(\cdot, t) : \Omega^0 \to \mathbb{R}^{2 \times 2}$ is the pull back of $\frac{\partial \phi^{-1}}{\partial x}(\cdot, t) : \Omega^t \to \mathbb{R}^{2 \times 2}$. We can show this by differentiating the relation

$$X = \phi^{-1}(\phi(X, t), t), \forall X \in \Omega^0, \forall t \geq 0$$

that is

$$I = \frac{\partial \phi^{-1}}{\partial x}(\phi(X, t), t) \frac{\partial \phi}{\partial X}(X, t)$$
where $I$ is the identity matrix. In index notation, this reads

$$\delta_{ij} = \frac{\partial \phi_i^{-1}}{\partial x_k}(\phi(X,t), t) \frac{\partial \phi_k}{\partial X_j}(X, t).$$

Now, since $\frac{\partial \phi}{\partial X}(X, t) = F(X, t)$ and $I = \frac{\partial \phi^{-1}}{\partial x}(\phi(X, t), t) \frac{\partial \phi}{\partial X}(X, t)$, $\frac{\partial \phi^{-1}}{\partial x}(\phi(X, t), t)$ must be the inverse of $F(X, t)$. In other words,

$$F^{-1}(X, t) = \frac{\partial \phi^{-1}}{\partial x}(\phi(X, t), t), \quad \forall X \in \Omega^0, \ t \geq 0.$$

I also sometimes use the notation

$$F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j} = \frac{\partial \phi_i^{-1}}{\partial x_j}.$$

Finally, of course this also implies that $\frac{\partial \phi^{-1}}{\partial x}(:, t) : \Omega^t \rightarrow \mathbb{R}^{2 \times 2}$ is the push forward of $F^{-1}(:, t) : \Omega^0 \rightarrow \mathbb{R}^{2 \times 2}$. 