Math 272A: Continuum Mechanics

1 Tensor calculus, Chapter 2 Gonzalez and Stuart (1/11/10):

2 Tensor calculus, Chapter 2 Gonzalez and Stuart (1/13/10):

- The Laplacian Δf of a function $f : \mathbb{E}^3 \rightarrow \mathbb{R}$ is defined as

  $$\Delta f = \nabla \cdot (\nabla f) : \mathbb{E}^3 \rightarrow \mathbb{R}.$$

  If we introduce the basis $\{\mathbf{e_i}\}$, then we can show $\nabla \cdot (\nabla f) = f_{,ii}$.

- Given $\mathbf{f} : \mathbb{E}^3 \rightarrow \mathrm{V}$, $\Delta \mathbf{f} : \mathbb{E}^3 \rightarrow \mathrm{V}$ is defined as

  $$\Delta \mathbf{f} = \nabla \cdot (\nabla \mathbf{f}) : \mathbb{E}^3 \rightarrow \mathrm{V}.$$

  If we introduce the basis $\{\mathbf{e_i}\}$, then we can show $[\nabla \cdot (\nabla \mathbf{f})]_i = f_{i,jj}$.

2.1 Integral theorems

We'll be interested in integrating over subsets B of Euclidean space $\mathbb{E}^3$. We won't need to let the class of B we are interested in be too esoteric. We'll restrict ourselves to regular subsets.

A subset B of $\mathbb{E}^3$ is **regular** if:

- It consists of finitely many open, disjoint and bounded sets.
- The bounding surface ∂B is piecewise smooth and consists of a finite number of disjoint components.
- Each component of ∂B is orientable (clearly has two sides). Roughly, this is regions of space with a finite number of holes or voids.

2.1.1 Divergence theorem

Given $\mathbf{v} : \mathbb{E}^3 \rightarrow \mathrm{V}$ and any regular $B \subset \mathbb{E}^3$

$$\int_{\partial B} \mathbf{v} \cdot \mathbf{n} \mathrm{dS}(\mathbf{x}) = \int_B \nabla \cdot \mathbf{v} \mathrm{d}\mathbf{x}.$$

With the introduction of a basis $\{\mathbf{e_i}\}$, this becomes

$$\int_{\partial B} v_i n_i \mathrm{dS}(\mathbf{x}) = \int_B v_{i,i} \mathrm{d}\mathbf{x}.$$

Given $\mathbf{S} : \mathbb{E}^3 \rightarrow \mathrm{V}^2$ and any regular $B \subset \mathbb{E}^3$

$$\int_{\partial B} \mathbf{S} \cdot \mathbf{n} \mathrm{dS}(\mathbf{x}) = \int_B \nabla \cdot \mathbf{S} \mathrm{d}\mathbf{x}.$$

With the introduction of a basis $\{\mathbf{e_i}\}$, this becomes

$$\int_{\partial B} S_{ij} n_j \mathrm{dS}(\mathbf{x}) = \int_B S_{ij,j} \mathrm{d}\mathbf{x}.$$

2.1.2 Stokes theorem

Given a surface $\Gamma \subset \mathbb{E}^3$ with boundary curve $C = \partial\Gamma$ and given $\mathbf{v} : \mathbb{E}^3 \rightarrow \mathrm{V}$ then

$$\int_\Gamma (\nabla \times \mathbf{v}) \cdot \mathbf{n} \mathrm{dS}(\mathbf{x}) = \int_C \mathbf{v} \cdot \mathbf{n} \mathrm{ds}.$$

2.1.3 Localization theorem

Given $B \subset \mathbb{E}^3$ and some $\phi : B \rightarrow \mathbb{R}$. If $\forall\Omega \subset,\ \Omega$ open

$$\int_\Omega \phi \mathrm{d}\mathbf{x} = 0$$

then $\phi(\mathbf{x}) = 0 \ \ \forall\mathbf{x}\in B$.

Proof is simple by contradiction, but will often just state, “by localization, ...”.