Homework 1: Due Feb. 22

1. 1.6 in Gonzalez and Stuart

2. 1.16 in Gonzalez and Stuart

3. Consider triangle $\mathbf{T} \subset \mathrm{E}^3$ and its image $\mathbf{t}$ under the second order tensor $\mathbf{F}$.
$$\mathbf{t} = \mathbf{F}\mathbf{T}$$
Define $d\mathbf{A}$ to be the area-weighted normal of triangle $\mathbf{T}$ (i.e. the normal to the triangle scaled by the area of the triangle). Write the area weighted normal $d\mathbf{a}$ to triangle $\mathbf{t}$ in terms of the $\mathbf{F}$ and $d\mathbf{A}$.

4. Show that if the scalar function $\Psi : V^2 \rightarrow \Re$ is isotropic, then the second order tensor fuction $\frac{\partial \Psi}{\partial \mathbf{F}} : V^2 \rightarrow V^2$ is also isotropic. That is, if $\Psi(\mathbf{F}) = \mathbf{\Psi}(\mathbf{Q}\mathbf{F}\mathbf{Q}^\mathbf{T})$ for all orthogonal $\mathbf{Q} \in V^\mathbf{2}$, then $\mathbf{Q}\frac{\partial \mathbf{\Psi}}{\partial \mathbf{F}}(\mathbf{F})\mathbf{Q}^\mathbf{T} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{F}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^\mathbf{T})$ for all orthogonal $\mathbf{Q} \in V^\mathbf{2}$.

5. We can define a scalar function Ψ of second order tensor argument $\mathbf{F}$ to be rotationally invariant if $\Psi(\mathbf{R}\mathbf{F}) = \mathbf{\Psi}(\mathbf{F})$ for all rotations $\mathbf{R}$. Show that this implies that $\frac{\partial \Psi}{\partial \mathbf{F}}(\mathbf{R}\mathbf{F}) = \mathbf{R}\frac{\partial \mathbf{\Psi}}{\partial \mathbf{F}}(\mathbf{F})$.

6. Show that if Ψ is isotropic and rotationally invariant that $\Psi(\mathbf{F}\mathbf{R}) = \mathbf{\Psi}(\mathbf{F})$, that $\Psi(\mathbf{F}) = \mathbf{\Psi}(\mathbf{U}\mathbf{F}\mathbf{V}^\mathbf{T})$ and that $\frac{\partial \Psi}{\partial \mathbf{F}}(\mathbf{U}\mathbf{F}\mathbf{V}^\mathbf{T}) = \mathbf{U}\frac{\partial \mathbf{\Psi}}{\partial \mathbf{F}}(\mathbf{F})\mathbf{V}^\mathbf{T}$ for all rotations $\mathbf{U}$ and $\mathbf{V}$. Also show that the above assumptions imply that when $\mathbf{F}$ is diagonal $\frac{\partial \Psi}{\partial \mathbf{F}}(\mathbf{F})$ is also diagonal.

7. Show that if $\Psi : V^2 \rightarrow \Re$ is isotropic as defined above that $\frac{\partial^2 \Psi}{\partial \mathbf{F}^\mathbf{2}} : V^2 \rightarrow V^4$ satisfies the following equality:
$$\mathbf{U}^\mathbf{T}\left(\frac{\partial^\mathbf{2} \mathbf{\Psi}}{\partial \mathbf{F}^\mathbf{2}}(\mathbf{F}) : \delta\mathbf{F}\right)\mathbf{V} = \frac{\partial^\mathbf{2} \mathbf{\Psi}}{\partial \mathbf{F}^\mathbf{2}}(\mathbf{U}^\mathbf{T}\mathbf{F}\mathbf{V}) : \left(\mathbf{U}^\mathbf{T}\delta\mathbf{F}\mathbf{V}\right)$$
for all $\mathbf{F}, \delta\mathbf{F} \in V^\mathbf{2}$ and for all rotations $\mathbf{U}, \mathbf{V} \in V^\mathbf{2}$.

8. Show that if $\mathbf{U}^\mathbf{T}\mathbf{F}\mathbf{V}$ is diagonal, that $\frac{\partial^2 \Psi}{\partial \mathbf{F}^\mathbf{2}}(\mathbf{U}^\mathbf{T}\mathbf{F}\mathbf{V})$ is block diagonal with one 3×3 block and three 2×2 blocks.

9. 3.6 in Gonzalez and Stuart

10. Prove that if the net body and surface forces on a region $B \subset \Omega$ is zero

$$\mathbf{r}[\mathbf{B}] = \int_{\mathbf{B}} \mathbf{bdx} + \int_{\partial\mathbf{B}} \sigma\mathbf{ndSx} = \mathbf{0}$$

then the net torque from those forces on the region

$$\tau[\mathbf{B}](\mathbf{z}) = \int_{\mathbf{B}} (\mathbf{x} - \mathbf{z}) \times \mathbf{bdx} + \int_{\partial\mathbf{B}} (\mathbf{x} - \mathbf{z}) \times \sigma\mathbf{ndSx}$$

is independent of the point $\mathbf{z}$ about which it is computed.

11. A small strain, linearly elastic, isotropic constitutive model relates stress to strain as

$$\sigma = \lambda \mathrm{Tr}(\epsilon)\mathbf{I} + \mathbf{2}\mu\epsilon$$

where ϵ is the small strain tensor defined in terms of the deformation gradient $\mathbf{F}$ as

$$\epsilon = \frac{1}{2}\left(\mathbf{F} + \mathbf{F^T}\right) - \mathbf{I}.$$

Show that $\sigma = \frac{\partial\Psi}{\partial\epsilon}(\epsilon)$ where

$$\Psi(\epsilon) = \frac{1}{2}\left(\lambda \mathrm{Tr}(\epsilon)^{\mathbf{2}} + \mathbf{2}\mu \mathrm{Tr}(\epsilon^{\mathbf{2}})\right)$$

Also show that if $\mathbf{F}$ is a rotation that

$$\Psi(\epsilon) = 16(\mu + \lambda) sin^4(\frac{\theta}{2})$$

where θ is the angle of rotation.

12. 4.22 in Gonzalez and Stuart