1. Write a program that computes the PLU of a matrix using partial pivoting. Test it on the matrix
\[
A = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 2 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & 0 \\
2 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

2. Derive an error bound for the PLU (with partial pivoting) and back substitution solution of a linear system. You can assume that PLU is backward stable and derive the bound in terms of the associated backward stability constants (and other factors).

3. Derive an error bound for the QR solution of the full rank least squares problem.

4. Investigation of QR solution of \(Ax = b\)
   (a) Modify the bound on the error introduced with the QR solution of a linear system to account for rounding error in representing the matrix \(A\) with floating point numbers.
   (b) Define the matrix \(A\) as
   \[
   A = QΛQ^T, \quad Λ = \begin{pmatrix}
κ & 0 \\
0 & 1
\end{pmatrix}, \quad Q = \begin{pmatrix}
\sqrt{\frac{2}{\kappa}} & 0 \\
0 & \sqrt{\frac{2}{\kappa}}
\end{pmatrix}.
   \]
   Show that this can be constructed without incurring any roundoff error (for certain choices of \(κ\)).
   (c) If
   \[
b = \begin{pmatrix}
κ \\
κ
\end{pmatrix}, \quad \text{then} \quad x = \begin{pmatrix}
1 \\
1
\end{pmatrix}.
   \]
   Use this to numerically investigate the constant arising from the backward stability property of the QR solution of the system. That is, we know \(\exists c\) such that
   \[
   (A + ΔA) \tilde{x} = b, \quad \frac{|ΔA|}{|A|} ≤ (c + O(\epsilon_{\text{machine}}^2)) \epsilon_{\text{machine}}
   \]
   where \(\tilde{x}\) is the QR algorithm approximation of \(x\). So, given that we know the influence of \(κ\) on the relative error \(\frac{\tilde{x} - x}{|x|}\), you should be able to estimate the constant numerically by plotting the relative error as a function of \(κ\).

5. Derive the eigenvalues of
\[
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots \\
0 & -1 & 2 & -1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 \\
2 & 0 & \cdots & 0 & 0 \end{pmatrix}.
\]
You can do this by looking at the action of the matrix on vectors \( x_j^k = e^{i\theta_k} \) for appropriate choice of \( \theta_k \).

6. Use the QR algorithm to compute the eigenvalues of the following two matrices.

(a) 

\[
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 \\
\end{pmatrix}
\]

(b) 

\[
A = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 2 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 & 0 \\
2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]