Homework 1.
Math 270B: Applied Numerical Linear Algebra

1. Determine the single precision (float) representations of $44$, $\frac{10}{3}$ and $-0.75$.

2. How many double precision numbers are between $e$ and $\pi$?

3. Prove that floating point division $(\div) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is backward stable over $\mathbb{R}$. You may assume that division returns
   \[ x \div y = \text{fl}(\text{fl}(x) \div \text{fl}(y)), \quad x \in \mathbb{R}, \ y \in \mathbb{R} \]

4. Is division well-conditioned?

5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Write the relative condition number of $f$ at $x \in \mathbb{R}^n$ in terms of the norm of the Jacobian of $f$ at $x$.

6. Consider the polynomial defined as $p(a_0, a_1; x) = x^2 + a_1 x + a_0$. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as
   \[ f \left( \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right) = \begin{pmatrix} \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \\ \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \end{pmatrix} \]
   You can think of $f$ as expressing the problem of root finding for the polynomial $p(a_0, a_1; x) = x^2 + a_1 x + a_0$ with coefficients $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$.
   (a) Show that the relative condition number will be unbounded for some $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$.
   (b) Write a program that demonstrates the consequences of this.

7. What is the smallest positive integer that does not belong to the single precision floating point numbers?

8. Write a program that demonstrates the ill-conditioning of subtraction.

9. Write a program that computes the Householder QR of a matrix. Test it on the matrix
   \[ A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \]

10. Consider the system
    \[ \begin{pmatrix} 1 & a \\ 1 & a \\ \vdots & \ddots \\ a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \]
    and the problem of finding $x$ that satisfies this. For what values of $a$ is the problem well-conditioned?